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A Test for Compactness of a Foliation

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ABSTRACT. We investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. We introduce the notion of the degree of compactness and prove a test for compactness.

In the present paper we investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. The problem of investigating the topological structure of level surfaces for such a form was posed by S. P. Novikov in [1]. This problem was treated in [2–5]. The present paper is devoted to the compactness problem for level surfaces. We introduce the notion of degree of compactness and prove a test for compactness expressed in the terms of the degree.

In §1 we give the necessary definitions and define the degree of compactness. The central result of the paper, i.e., the test for compactness of a foliation, is proved in §2. In §3 we present some consequences: a relation between the degree of compactness and the degree of irrationality of the form, and a more detailed investigation of the two-dimensional case.

The present paper is a natural continuation of [6].

§1. Preliminary definitions

Consider a smooth compact n-dimensional manifold M and a closed 1-form ω on M with nondegenerate isolated singularities.

Definition 1 [7]. A point $p \in M$ is said to be a *regular singularity* of the differential form ω if in a neighborhood $O(p)$ we have $\omega = df$, where f is a Morse function with a singularity at p. There exist, therefore, coordinates $x^1, \dots, x^n$ such that in this neighborhood we have

$$\omega = \sum_{i=1}^{k} x_i dx^i - \sum_{i=k+1}^{n} x_i dx^i.$$

The number $\min(k, n-k)$ is called the *index* of the singular point.

On the set $M - \operatorname{Sing}\omega$ the form ω determines a foliation $\mathcal{F}_\omega$ of codimension 1. If the index of the singular point P is equal to zero, then there exists a foliation of a neighborhood of P into spheres. If $\operatorname{ind} P = 1$, then there exists a fiber that becomes locally arcwise connected after adding the singular point P. This fiber is called the *canonical fiber*. For $\operatorname{ind} P > 1$ all the fibers in the neighborhood of P are locally arcwise connected.

The foliation $\mathcal{F}_\omega$ contains fibers of three kinds [7]:

1) compact fibers admitting a neighborhood consisting of diffeomorphic fibers;
2) conic fibers, i. e., fibers that may be made locally arcwise connected in a neighborhood of a singular point by adding this singular point to the fiber;
3) all the other noncompact fibers.

Below we assume that the singular point P belongs to the fiber, and thus all the fibers are arcwise connected.

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Definition 2. Consider a compact fiber γ of $\mathcal{F}_\omega$ and the mapping $\gamma \mapsto [\gamma] \in H_{n-1}(M)$. Then the images of all compact fibers are spanned by a subgroup in $H_{n-1}(M)$. We denote this subgroup by H_ω and call $\operatorname{rk} H_\omega$ the *degree of compactness* of the foliation $\mathcal{F}_\omega$.

Since $H_\omega \subseteq H_{n-1}(M)$, we obtain $0 \le \operatorname{rk} H_\omega \le \beta_{n-1}$, where $\beta_{n-1} = \operatorname{rk} H_{n-1}(M)$. If all fibers of $\mathcal{F}_\omega$ are noncompact, then $\operatorname{rk} H_\omega = 0$. The converse is false: some compact fibers may prove to be homologous to zero. Moreover, there exist compact foliations with $\operatorname{rk} H_\omega = 0$. To obtain such a foliation, it is sufficient to consider a manifold M such that $H_{n-1}(M) = 0$. The foliation associated with any closed form is obviously compact, and all the fibers are homologous to zero.

Consider the group $H_{n-1}(M)$ and the intersection map

$$\circ : H_{n-1}(M) \times H_{n-1}(M) \to H_{n-2}(M)$$

for homology classes, which is defined in the following way [8]. Let x, $y \in H_{n-1}(M)$, and let D denote Poincaré duality; then $x \circ y = Dx \cap y$. If the homology classes x and y are realized by the submanifolds X and Y, then $x \circ y$ is the homology class of $X \cap Y$. The intersection is skew-symmetric: $x \circ y = -y \circ x$.

Definition 3. Consider the subgroup $H \subset H_{n-1}(M)$ such that for all $x, y \in H$ we have $x \circ y = 0$. We call H the *isotropic subgroup* with respect to the intersection of cycles. An isotropic subgroup H is called *maximal*, if for all $x \in H$, $x \neq 0$ and $y \notin H$ we have $x \circ y \neq 0$.

The subgroup H_ω of compact fibers obviously is an isotropic subgroup in $H_{n-1}(M)$.

Denote by M_ω the set obtained by eliminating from M all the maximal neighborhoods consisting of diffeomorphic compact fibers and all the fibers that may be compactified by adding singular points.

§2. The main theorem

Let us establish the validity of the following test.

Theorem. *If the subgroup H_ω spanning all the compact fibers is a maximal isotropic subgroup of the homology group $H_{n-1}(M)$, then $M_\omega = \varnothing$.*

Proof. Suppose the subgroup H_ω has maximal rank, and let $H_\omega = \langle [\gamma_1], \dots, [\gamma_N] \rangle$, where γ_i are fibers of $\mathcal{F}_\omega$. Cutting M along the fibers $\gamma_1, \dots, \gamma_n$, we obtain a manifold M' with boundary.

Let $\varphi \colon M' \to M$ be the gluing map, and let $i \colon \partial M' \to M'$ be the boundary inclusion mapping.

Lemma 1. *If H_ω is a maximal isotropic subgroup, then the mapping $i_* \colon H_{n-1}(\partial M') \to H_{n-1}(M')$ is surjective.*

Proof. The gluing map $\varphi \colon M' \to M$ induces the mapping of pairs $\varphi \colon (M', \partial M') \to (M, \bigcup \gamma_i)$. Let us set $\varphi|_{\partial M'} = \varphi_1$ and consider the commutative diagram

$$\begin{array}{ccc} H_{n-1}(\partial M') & \xrightarrow{i_*} & H_{n-1}(M') \\ \Big\downarrow{\scriptstyle \varphi_{1*}} & & \Big\downarrow{\scriptstyle \varphi_*} \\ H_{n-1}(\bigcup \gamma_i) & \xrightarrow{j} & H_{n-1}(M). \end{array}$$

We claim that 1) the mapping j is injective, 2) the mapping φ_{1*} is surjective, 3) $\ker \varphi_* \subset \operatorname{im} i_*$.

1) Since the fibers γ_i do not intersect, $\gamma_i \cap \gamma_j = \varnothing$, the Mayer–Vietoris exact sequence gives

$$H_{n-1}(\bigcup \gamma_i) = \oplus H_{n-1}(\gamma_i).$$

By assumption, the cycles $[\gamma_i]$ are independent in M, and, therefore,

$$\oplus H_{n-1}(\gamma_i) = \langle [\gamma_1], \dots, [\gamma_N] \rangle = H_\omega$$

and the mapping $j\colon H_\omega \to H_{n-1}(M)$ is an inclusion.

2) Since $\varphi(\partial M') = \bigcup \gamma_i$, it follows that the mapping φ_{1*} is surjective.

3) If $\varphi_* z' = 0$, then $\varphi(z') = \partial S$, where $S \subset M$. After the cutting process S is bounded by z' and the γ_i that satisfy $S \cap \gamma_i \neq \varnothing$. Thus, $z' \in \operatorname{im} i_*$.

Consider $z' \in H_{n-1}(M')$, then $z' \cap \partial M' = \varnothing$; hence, $\varphi(z') \cap \gamma_i = \varnothing$, $i = 1, \dots, N$, and so $\varphi_* z' \circ \gamma_i = 0$, $i = 1, \dots, N$. By assumption H_ω is maximal and, therefore, $\varphi_* z' \in H_\omega$. Since $j\colon H_\omega \to H_{n-1}(M)$ is an inclusion, there exists an element $z \in j^{-1}(\varphi_* z')$. The mapping φ_{1*} is surjective, and hence the preimage of z in $H_{n-1}(\partial M')$ exists: $z_0 = \varphi_*^{-1}(z)$. The commutativity of the diagram then implies

$$\varphi_* i_* z_0 = j \varphi_{1*} z_0 = j(z) = \varphi_* z'.$$

Thus, $z' - i_* z_0 \in \ker \varphi_*$. Due to the reasoning above, $\ker \varphi_* \subset \operatorname{im} i_*$; therefore, $z' - i_* z_0 \in \operatorname{im} i_*$ and $z' \in \operatorname{im} i_*$. It follows then that i_* is surjective. Lemma 1 is proved. $\square$

To the short exact sequence

$$0 \to Z \to H_{n-1}(\partial M') \xrightarrow{i_*} H_{n-1}(M') \to 0,$$

where $Z = \ker i_*$, let us apply the functor $\otimes \mathbb{R}$, which is covariant and right exact [8]. We obtain the exact sequence

$$Z \otimes \mathbb{R} \to H_{n-1}(\partial M') \otimes \mathbb{R} \xrightarrow{i_R} H_{n-1}(M') \otimes \mathbb{R} \to 0,$$

where the mapping i_R is surjective as well.

According to the universal coefficients theorem, we have $H_k(M, \mathbb{R}) = H_k(M) \otimes \mathbb{R}$. Thus, Lemma 1 implies that for homology with coefficients in $\mathbb{R}$ the mapping $i_*\colon H_{n-1}(\partial M', \mathbb{R}) \to H_{n-1}(M', \mathbb{R})$ is surjective as well.

Lemma 2. *Let $i\colon \partial M' \to M$ be the inclusion mapping. If the mapping*

$$i_*\colon H_{n-1}(\partial M', \mathbb{R}) \to H_{n-1}(M', \mathbb{R})$$

is surjective, then the mapping

$$j\colon H_1(\partial M', \mathbb{R}) \to H_1(M', \mathbb{R})$$

is surjective as well.

Proof. Consider the commutative diagram (all the homology groups have coefficients in $\mathbb{R}$):

$$\begin{array}{ccc} H_1(M', \partial M') & \xrightarrow{\partial} & H_0(\partial M') \\ \Big\downarrow D & & \Big\downarrow D \\ H^{n-1}(M', \mathbb{R}) & \xrightarrow{i_*} & H^{n-1}(\partial M', \mathbb{R}). \end{array} \tag{1}$$

Here D is Poincaré duality for manifolds with boundary. By assumption, the map $i_*\colon H_{n-1}(\partial M', \mathbb{R}) \to H_{n-1}(M', \mathbb{R})$ is surjective. Consider the short exact sequence

$$0 \to Z \to H_{n-1}(\partial M', \mathbb{R}) \xrightarrow{i_*} H_{n-1}(M', \mathbb{R}) \to 0,$$

where $Z = \ker i_*$. Since

$$H^{n-1}(\partial M', \mathbb{R}) = \operatorname{Hom}(H_{n-1}(\partial M', \mathbb{R}), \mathbb{R}) \quad \text{and} \quad H^{n-1}(M', \mathbb{R}) = \operatorname{Hom}(H_{n-1}(M', \mathbb{R}), \mathbb{R}),$$

to this sequence we may apply the functor $\operatorname{Hom}(\ ,\mathbb{R})$, which is contravariant and right exact [8]. We obtain the following exact sequence:

$$\operatorname{Hom}(Z, \mathbb{R}) \leftarrow \operatorname{Hom}(H_{n-1}(\partial M', \mathbb{R}), \mathbb{R}) \xleftarrow{i_*} \operatorname{Hom}(H_{n-1}(M', \mathbb{R}), \mathbb{R}) \leftarrow 0,$$

and i_* is therefore injective. Then in diagram (1), we have $\ker \partial = 0$.

Consider the exact homology sequence of the pair $(M', \partial M')$

$$\to H_1(\partial M') \xrightarrow{j} H_1(M') \xrightarrow{l} H_1(M', \partial M') \xrightarrow{\partial} H_0(\partial M') \to .$$

Since $\ker \partial = \operatorname{im} l = 0$, we obtain $\ker l = H_1(M')$. Since $\operatorname{im} j = \ker l = H_1(M')$, the mapping j is surjective. Lemma 2 is proved. $\square$

Let us consider the mapping $\varphi\colon M' \to M$, and set $\omega' = \varphi^*\omega$, which will be the restriction of ω to M'. For all $z' \in H_1(M')$ we have

$$\int_{z'} \omega' = \int_{z'} \varphi^*\omega = \int_{\varphi_* z'} \omega.$$

By Lemma 2, $z' \in H_1(\partial M')$, and therefore $\varphi(z') \in \bigcup \gamma_i$. Then $\int_{\varphi(z')} \omega = 0$. Thus,

$$\int_{z'} \omega' = 0 \quad \forall z' \in H_1(M'),$$

and, since ω is a Morse form on M', the foliation is compact. It follows then that the foliation on M is also compact. The theorem is proved. □

Remark. The converse is not valid for dimensions greater than 2. Thus, there exists a compact foliation on $\mathbb{S}^2 \times \mathbb{S}^1$ with all the fibers homologous to zero, i.e., the subgroup H_ω is not a maximal subgroup.

§3. Some corollaries

The rank of a maximal isotropic subgroup in $H_{n-1}(M)$ is an attribute of the manifold M; it does not depend on any foliation. Denote this number by h_0. In the two-dimensional case, the rank of a maximal isotropic subgroup is equal to the number of handles of the manifold, $h_0(M_g^2) = g$. The following criterion has been established in [6]: *a foliation $\mathcal{F}_\omega$ on M_g^2 is compact iff* $\operatorname{rk} H_\omega = g$. The compactness degree of the foliation associated with a Morse form is related to the degree of irrationality of the form ω. For example, on M_g^2 the inequality $\operatorname{dirr}\omega + \operatorname{rk} H_\omega \le 2g - 1$ is valid.

Corollary 1. *Let ω be a closed* 1*-form with Morse singularities determining on a compact manifold M^n the foliation $\mathcal{F}_\omega$. If H_ω is a maximal isotropic subgroup, then* $\operatorname{dirr}\omega < \operatorname{rk} H_\omega$.

Proof. Indeed, if H_ω is a maximal isotropic subgroup, i.e., $\operatorname{rk} H_\omega = h_0(M)$, then the proof of the theorem above implies that the 1-form ω may have nontrivial integrals only over cycles transversal to the fibers γ_k. Therefore, $\operatorname{dirr}\omega \le \operatorname{rk} H_\omega - 1 = h_0(M) - 1$. Corollary 1 is proved. □

Corollary 2. *If* $\operatorname{dirr}\omega \ge g$, *then the foliation of a Morse form ω on M^2 has a noncompact fiber.*

Indeed, by the compactness criterion for a foliation on M_g^2, if $\mathcal{F}_\omega$ is compact, then H_ω is maximal. The previous statement then implies that $\operatorname{dirr}\omega \le h_0(M_g^2) - 1$. Corollary 2 is proved.

References

1. S. P. Novikov, "The Hamiltonian formalism and a multivalued version of Morse theory," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **37**, No. 5, 3–49 (1982).
2. S. P. Novikov, "Critical points and level surfaces of multivalued functions," in: *Trudy Mat. Inst. Steklov* [*Proc. Steklov Inst. Math.*], **166**, 201–209 (1984).
3. A. V. Zorich, "Quasiperiodic structure of level surfaces of a 1-form close to a rational one; Novikov's problem," *Izv. Akad. Nauk SSSR, Ser. Mat.* [*Math. USSR-Izv.*], **51**, No. 6, 1322–1344 (1987).
4. T. K. T. Le, "On the structure of level surfaces of a Morse form," *Mat. Zametki.* [*Math. Notes*], **44**, No. 1, 124–133 (1988).
5. L. A. Alaniya, "On the topological structure of the level surfaces of Morse 1-forms," *Uspekhi Mat. Nauk* [*Russian Math. Surveys*], **46**, No. 3, 179–180 (1991).
6. I. A. Mel'nikova, "A noncompactness test for a foliation on M_g^2," *Mat. Zametki.* [*Math. Notes*], **53**, No. 3, 158–160 (1993).
7. Henc Damir, "Ergodicity of foliations with singularities," *J. Funct. Anal.*, **75**, No. 2 (1987).
8. A. Dold, *Lectures on Algebraic Topology*, 2nd ed., Berlin, (1980).

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