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NUMBER OF MINIMAL COMPONENTS AND HOMOLOGICALLY INDEPENDENT COMPACT LEAVES FOR A MORSE FORM FOLIATION

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Abstract

The numbers $m(\omega)$ of minimal components and $c(\omega)$ of homologically independent compact leaves of the foliation of a Morse form ω on a connected smooth closed oriented manifold M are studied in terms of the first non-commutative Betti number $b_1'(M)$. A sharp estimate $0 \leqq m(\omega) + c(\omega) \leqq b_1'(M)$ is given. It is shown that all values of $m(\omega) + c(\omega)$, and in some cases all combinations of $m(\omega)$ and $c(\omega)$ with this condition, are reached on a given M. The corresponding issues are also studied in the classes of generic forms and compactifiable foliations.

1. Introduction and announce of the results

Consider a connected closed oriented manifold M with a Morse form ω, i.e., a closed 1-form with Morse singularities $\operatorname{Sing}\omega$ (locally the differential of a Morse function). This form defines a foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$.

The number $m(\omega)$ of minimal components and $c(\omega)$ of homologically independent compact leaves are important topological characteristics of the foliation. For example, if $\mathcal{F}_\omega$ is compactifiable, i.e. $m(\omega) = 0$, then $\operatorname{rk}\omega \leqq c(\omega)$, where $\operatorname{rk}\omega$ is the number of its incommensurable periods; for the the cycle rank $m(\Gamma)$ of the foliation graph Γ it holds $m(\Gamma) = c(\omega)$ (Section 2.1; [4]).

Considerable effort has been devoted to estimating these numbers. Obviously, $c(\omega) \leqq b_1(M)$, where $b_1(M)$ is the Betti number; in [1] ($\dim M \geqq 3$) and [7] (M_g^2) it was shown that $2m(\omega) \leqq b_1(M)$. In [4] these facts were

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combined into

$$0 \leqq c(\omega) + 2m(\omega) \leqq b_1(M). \tag{1}$$

In [11] it was shown that $c(\omega) \leqq h(M)$, where $h(M) \leqq b_1(M)$ is another homological characteristic of the manifold; in [4] this was generalized to an independent estimate

$$0 \leqq c(\omega) + m(\omega) \leqq h(M).$$

An independent estimate in terms of $\operatorname{Sing} \omega$ was given in [12]:

$$0 \leqq c(\omega) + m(\omega) \leqq \frac{|\Omega_1| - |\Omega_0|}{2} + 1,$$

where Ω_1 is the set of conic singularities and Ω_0 of centers. These estimates were, though, not exact.

In this paper we give an exact estimate in terms of the non-commutative Betti number $b_1'(M)$ – the maximal rank of a free quotient group of the fundamental group $\pi_1(M)$ [9]; obviously $b_1'(M) \leqq b_1(M)$ and as we show, $b_1'(M) \leqq h(M)$. We prove (Theorem 3) that

$$0 \leqq c(\omega) + m(\omega) \leqq b_1'(M) \tag{2}$$

and show that all intermediate values are reached on M even for $c(\omega)$ alone:

$$0 \leqq c(\omega) \leqq b_1'(M), \tag{3}$$

and even in the class of compactifiable foliations (Theorem 8). In particular, on any M there exists a compactifiable foliations with all (compact) leaves being homologically trivial; such forms are exact (Theorem 4). On M_g^2, all combinations of $c(\omega)$ and $m(\omega)$ that satisfy (2) are reached (Propositon 7); $b_1'(M_g^2) = g$ (Lemma 2). Possibly all combinations of $c(\omega)$ and $m(\omega)$ that satisfy both (1) and (2) are reached on a given manifold (Conjecture 11); these conditions are independent if $\dim M \geqq 3$ (Remark 9, Example 10).

A Morse form is called generic if each its singular leaf contains a unique singularity [2]; such forms are dense in the space of Morse forms. All statements mentioned above hold in the class of generic forms, with some exceptions for M_g^2 (Remark 12). Specifically, the exact lower bound in (2) on M_g^2 except for S^2 is 1 (Proposition 14):

$$1 \leqq c(\omega) + m(\omega) \leqq b_1'(M_g^2) = g$$

and for compactifiable foliations of generic forms on M_g^2, (3) is reduced to $c(\omega) = g$ (Lemma 13, Remark 16). With this, for generic forms on M_g^2 possible are all combinations of $c(\omega)$ and $m(\omega)$ such that if $\mathcal{F}_\omega$ is compactifiable then $c(\omega) = g$, otherwise $1 \leqq c(\omega) + m(\omega) \leqq g$ (Proposition 17).

The paper is organized as follows. In Section 2 we give necessary definitions and prove some useful facts. In Section 3 we prove the main inequality (2). In Section 4 we prove the exactness of this inequality by constructing forms with extremal values. In Section 5 we show that all intermediate values within the bounds (2) are reached, and in some cases all values of $c(\omega)$ and/or $m(\omega)$ allowed by (2) are reached (this does not eliminate the simpler Section 4 since its examples are used as building blocks). Finally, in Section 6 we give analogs of our most important statements for the class of generic forms.

2. Definitions and useful facts

In this paper, M is a connected closed oriented manifold. A closed 1-form ω on M is called a *Morse form* if it is locally the differential of a Morse function. The set $\operatorname{Sing}\omega = \{p \in M \mid \omega(p) = 0\}$ of its singularities is finite, since they are isolated and M is compact. In this paper we consider only *singular* forms, i.e., $\operatorname{Sing}\omega \neq \emptyset$. On $M \setminus \operatorname{Sing}\omega$ the form ω defines a foliation $\mathcal{F}_\omega$.

2.1. Leaves and components; $c(\omega)$ and $m(\omega)$

A leaf $\gamma \in \mathcal{F}_\omega$ is called *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact; otherwise it is called *non-compactifiable.* A foliation is called *compactifiable* if all its leaves are compactifiable. The number of non-compact compactifiable leaves γ_i is finite, since each singularity can compactify no more than four leaves.

A *singular* leaf γ^0 is a maximal union of one or more leaves and one or more singularities such that for any two points $p, q \in \gamma^0$ there exists a path $\alpha\colon [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega\big(\dot\alpha(t)\big) = 0$ for all t.

A Morse form (or function) is called *generic* if each its singular leaf contains a unique singularity [2]. Generic forms are dense in the space of Morse forms.

By $m(\omega)$ we denote the number of minimal components of $\mathcal{F}_\omega$. A *minimal component* is a connected component of the union of non-compactifiable leaves. The latter union is open, the number of minimal components is finite, and each non-compactifiable leaf is dense in its minimal component [1, 6]. Obviously, $\mathcal{F}_\omega$ is *compactifiable* if $m(\omega) = 0$.

LEMMA 1. *On M_g^2, a minimal component contains two cycles z, z' such that $z \cdot z' \neq 0$.*

PROOF. Let U be a minimal component and $s \subset U$ a curve such that $\int_s \omega \neq 0$. Consider the cycle in $H_1(\overline{U}, \partial\overline{U})$ corresponding to $[s] \in H_1(U)$. By Poincaré duality it defines a non-zero cocycle $\alpha \in H^1(U, \mathbb{Z})$. Since torsion $\big(H_1(M_g^2)\big) = 0$, $[s]$ can be viewed as an element of Hom $\big(H_1(U), \mathbb{Z}\big)$, i.e. $\alpha(z) = [s] \cdot z$. Since $\alpha \neq 0$ there exists $z \in H_1(U)$ such that $[s] \cdot z \neq 0$. □

By $c(\omega)$ we denote the number of homologically independent compact leaves of $\mathcal{F}_\omega$. For a compact leaf γ there exists an open neighborhood consisting solely of compact leaves: indeed, integratingω gives a function f with $df = \omega$ near γ; hence the union of all compact leaves is open.

A connected component of the union of compact leaves of$\mathcal{F}_\omega$ is called a *maximal component.* Since $\operatorname{Sing}\omega \neq \emptyset$, it is a (maximal) cylindrical neighborhood $\gamma \times (0,1)$ of any its leaf $\gamma \in \mathcal{F}_\omega$ and consists of compact leaves diffeomorphic to γ. Its boundary is a union of some non-compact compactifiable leaves and singularities. Obviously, the number of maximal components is finite [4].

The *foliation graph* Γ is the graph whose edges are maximal components (their boundary has one or two connected components) and vertices are connected components of the union of all non-compact leaves, i.e., a vertex consists of singularities, singular leaves, and/or minimal components; an edge is incident to a vertice if they adjoin in M. The structure of the foliation graph closely reflects that of the foliation itself; see details in [4]. In particular,

$$m(\Gamma) = c(\omega), \tag{4}$$

where $m(\Gamma)$ is the cycle rank [5] of the graph.

By $\operatorname{rk}\omega$ we denote the number of incommensurable periods of the formω, i.e., $\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}}\big\{\int_{z_1}\omega, \dots, \int_{z_k}\omega\big\}$, where $z_1, \dots, z_k$ is a basis of $H_1(M)$. If $\mathcal{F}_\omega$ is compactifiable then

$$\operatorname{rk}\omega \leqq c(\omega); \tag{5}$$

in particular, $c(\omega) = m(\omega) = 0$ implies $\omega = df$ [4].

2.2. Non-commutative Betti number $b_1'(M)$

By $b_1'(M)$ we denote the non-commutative Betti number – the maximal rank (number of free generators) of a free quotient group of$\pi_1(M)$ [1]; $b_1'(M) \leqq b_1(M)$, the Betti number [9].

LEMMA 2. $b_1'(M_g^2) = g$.

PROOF. Let $M = M_g^2$. Obviously, $b_1'(M) \geqq g$ since the fundamental group

$$\pi_1(M_g^2) = \langle a_i, b_i,\ i = 1, \dots,\ g \mid a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} = 1\rangle$$

can be mapped onto a free subgroup $\langle a_i,\ i = 1, \dots, g\rangle$. Let us show $b_1'(M) \leqq g$.

Given a surjection $\pi_1(M) \to F$, $\operatorname{rk} F = b_1'(M)$, consider a continuous map $f : M \to W$, $W = \bigvee_{i=1}^{b_1'(M)} S_i^1$. Let $p_i \in S_i^1$ be its regular values; $c_i = f^{-1}(p_i)$ are circles in M. Consider the map $f_* : H_1(M) \to H_1(W)$. Cycles $z_i \in H_1(M)$ such that $f_* z_i = [S_i^1] \in H_1(W)$ are independent. By construction $[c_i] \cdot z_j = \delta_{ij}$, therefore $[c_i]$, $i = 1, \dots, b_1'(M)$, are also independent in $H_1(M)$. Since $[c_i] \cdot [c_j] = 0$, we obtain $b_1'(M) \leqq g$. □

A Morse form (or a minimal component) is called *weakly complete* if it has no centers and any its singular leaf containing a conic singularity (of index 1 or $n-1$) stays connected after removal this singularity. In any nonzero cohomology class there exists a weakly complete Morse form [8].

3. Main theorem: bounds on $c(\omega) + m(\omega)$

THEOREM 3. *Let M be a smooth closed oriented manifold and ω a Morse form on it. Then*

$$0 \leqq c(\omega) + m(\omega) \leqq b_1'(M) \tag{6}$$

and all intermediate values are reached on a given M; in particular, the bounds are exact.

PROOF. (i) $\dim M \geqq 3$. Let $\mathcal{F}_\omega$ contain m_1 not weakly complete and m_2 weakly complete minimal components, $m_1 + m_2 = m(\omega)$. By [9, Theorem I.1] the fundamental group of the space of leaves $\pi_1(M/\omega)$ can be represented as a free product of free abelian groups

$$\pi_1(M/\omega) = (\underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k_0}) * (\underbrace{\mathbb{Z} * \dots * \mathbb{Z}}_{k_1}) * (P_1 * \dots * P_{m_2}),$$

where the first k_0 factors correspond to the set of the compact leaves and form $\pi_1(\Gamma)$ (Γ is the foliation graph); the next k_1 factors correspond to the set of weakly complete minimal components, $k_1 \geqq m_1$; and the groups P_i correspond to weakly complete minimal components, $\operatorname{rk} P_i \geqq 2$, with $k_0 + k_1 + m_2 \leqq b_1'(M)$.

Since $k_0 = m(\Gamma)$, the latter inequality and (4) implies (6).

Erratum:

k1 factors correspond to the set of not weakly complete minimal components (*ii*) $\dim M = 2$. Let $\gamma_1, \ldots, \gamma_c$, $c = c(\omega)$, be homologically independent compact leaves and $U_1, \ldots, U_m$, $m = m(\omega)$, minimal components of $\mathcal{F}_\omega$. By Lemma 1 there exist $z_i, z_i' \subset U_i$ such that $z_i \cdot z_i' \neq 0$. The cycles $[\gamma_1], \ldots, [\gamma_c]$, $z_1, \ldots, z_m$ are independent; indeed,

$$\left(\sum_{i=1}^{c} n_i[\gamma_i] + \sum_{i=1}^{m} m_i z_i \right) \cdot z_j' = 0$$

implies all $n_i, m_i = 0$. Moreover, all $[\gamma_i] \cdot [\gamma_j] = [\gamma_i] \cdot z_j = z_i \cdot z_j = 0$. Thus $c + m \leqq g = b_1'(M_g^2)$ (by Lemma 2).

Existence of all values within the bounds (6) follows from Theorem 8 below. Exactness of the bounds also independently follows from Theorem 4 and Proposition 5. □

4. Existence of extremal values of $c(\omega)$ and $m(\omega)$

THEOREM 4. *On M there exists a Morse form ω with $c(\omega) = m(\omega) = 0$, i.e., $\mathcal{F}_\omega$ being compactifiable and all its leaves homologically trivial (such forms are exact).*

PROOF. Exactness of the form follows from (5). We will construct a Morse function f with $c(df) + m(df) = 0$.

(*i*) $\dim M \geqq 3$. Consider a tubular neighborhood Y of a wedge sum $\bigvee_{i=1}^{b_1(M)} S_i^1$ of circles that generate a basis of $H_1(M)$; ∂Y is connected and homologically trivial. Let ∂Y be a leaf of f.

The inside of Y can be foliated as shown in Fig. 1. The figure shows the neighborhood of a wedge sum of (two) circles S_i^1 (the edges of the cylinders are identified). Take a center p_0; surrounding leaves are spheres. Extend them along S_1^1 until they self-intersect forming a conic singularity p_1 and then an $S^1 \times S^{n-2}$. Extend the latter along S_2^1 until it self-intersects forming a conic singularity p_2. Repeating this for all S_i^1 will foliate Y such that all leaves are homologically trivial and ∂Y is a leaf.

Now extend f on the rest of M; all its leaves are homologically trivial. Indeed, denote $M' = \overline{M \setminus Y}$; $\partial M' = \partial Y$. By construction, $H_1(M', \partial M') = 0$, then

$$H^{n-1}(M', \mathbb{Z}) = H_{n-1}(M') \oplus \operatorname{torsion}\big(H_{n-2}(M')\big) = 0 \tag{7}$$

by the Poincaré duality. We obtain $H_{n-1}(M') = 0$.

(*ii*) $\dim M = 2$. On S^2 all leaves are homologically trivial. Let $M = M_g^2$, $g \geqq 1$. Fig. 2(a) shows a torus T^2 (the opposite sides of the square are

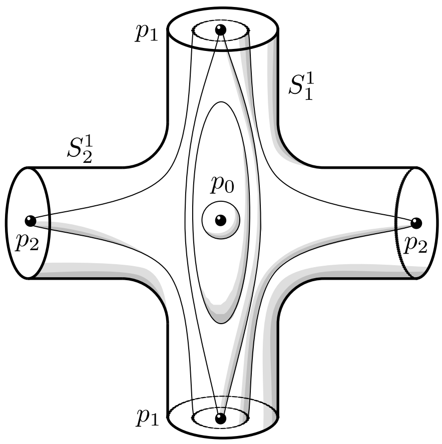

Fig. 1. Foliating the inside of Y

identified) with a desired foliation: p_i are centers and q_i saddles. Finally, $M_g^2 = \sharp_{i=1}^g T_i^2$ is assembled as a connected sum of tori, see Fig. 2(*b*): a leaf surrounding p_2 of each previous torus is identified with a leaf surrounding p_1 of the next torus. □

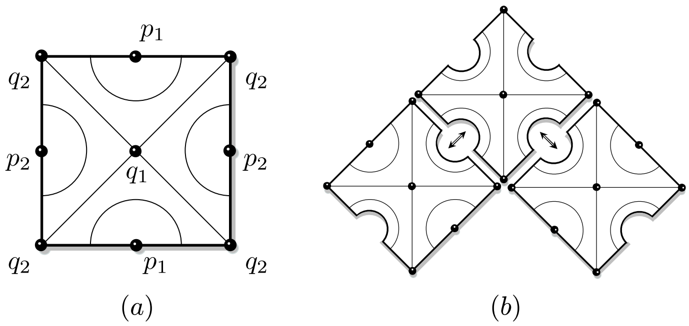

Fig. 2. Compactifiable foliation with $c(\omega) = 0$ on (*a*) T^2, (*b*) $M_g^2 = \sharp\, T_i^2$

Proposition 5. *On M there exists a Morse form ω with $c(\omega) = b_1'(M)$ and $m(\omega) = 0$ ($\mathcal{F}_\omega$ compactifiable).*

Proof. By definition of $b_1'(M)$ there exists a surjective homomorphism $\pi_1(M) \to F$, where F is a free group, $\operatorname{rk} F = b_1'(M)$. Consider a corresponding map $\varphi : M \to W$, where $W = \bigvee_{i=1}^{b_1'(M)} S_i^1$. Let $\alpha_W \in H^1(W, \mathbb{R})$, $\operatorname{rk} \alpha_W = b_1'(M)$, and $\alpha = \varphi^* \alpha_W$.

Let $x_i \in S_i^1$ be regular values of φ; each $M_i = \varphi^{-1}(x_i)$ is a compact submanifold of M. Denote by M' the result of cutting M open along the M_i;

$\partial M' = \bigcup_i (M_i^+ \cup M_i^-)$. We obtain $\alpha|_{M'} = 0$. Thus we can choose on M' a Morse function f without singularities on $\partial M'$ such that it is constant on each connected component of $\partial M'$, $f(M_i^+) - f(M_i^-) = \int_{S_i^1} \alpha$, and $f|_{\partial M'}$ fits together smoothly, giving on M a Morse form $\omega \sim \alpha$. Obviously, $\mathcal{F}_\omega$ is compactifiable; thus by (5) it holds $c(\omega) \geqq \operatorname{rk} \omega = b_1'(M)$. From Theorem 3 it follows $c(\omega) = b_1'(M)$ and $m(\omega) = 0$. □

PROPOSITION 6. *If $b_1(M) \geqq 2$ then on M there exists a Morse form ω with minimal foliation; in particular, $c(\omega) = 0$ and $m(\omega) = 1$.*

PROOF. For $\dim M \geqq 3$ this was proved in [1]. A corresponding foliation on $M_g^2 = \sharp T_i^2$ is shown in Fig. 3. □

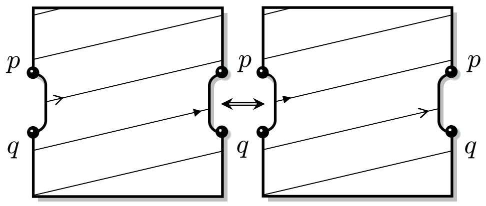

Fig. 3. Minimal foliation on $M_g^2 = \sharp(T_i^2)$

5. Existence of intermediate values of $c(\omega)$ and $m(\omega)$

PROPOSITION 7. *Let $c, m \in \mathbb{Z}$. On M_g^2 there exists a Morse form ω such that $c(\omega) = c$ and $m(\omega) = m$ iff*

$$0 \leqq c + m \leqq b_1'(M_g^2) = g.$$

PROOF. By Theorem 3 and Lemma 2, we only need to show existence. To construct the desired ω represent M_g^2 as a connected sum of c tori with a compact, and m with a minimal, non-singular foliation plus an M_{g-c-m}^2 foliated as in Theorem 4, glued together by a circle inserted between leaves via a saddle as shown in Fig. 4. □

THEOREM 8. *Let $c \in \mathbb{Z}$. On M there exists a Morse form ω with $c(\omega) = c$ iff*

$$0 \leqq c \leqq b_1'(M).$$

The form can be chosen with $m(\omega) = 0$ ($\mathcal{F}_\omega$ compactifiable).

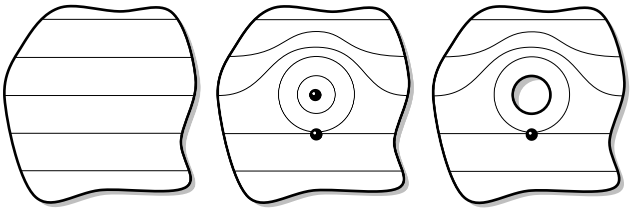

Fig. 4. Preparing a summand for the connected sum

PROOF. By Theorem 3 we only need to show existence. For $\dim M = 2$ see Proposition 7; let $\dim M \geqq 3$. By Proposition 5, on M there exists a Morse form ω_0 with compactifiable foliation and $c(\omega_0) = b_1'(M)$. Starting from this foliation, we will construct a compactifiable foliation with $c(\omega) = c$.

Let $\gamma_1, \ldots, \gamma_c$ be homologically independent compact leaves of $\mathcal{F}_{\omega_0}$. Denote by $\mathcal{M}$ the result of cutting M open along γ_i; $\partial\mathcal{M} = \bigcup_i (\gamma_i^+ \cup \gamma_i^-)$. We will construct on $\mathcal{M}$ a form ω with no homologically non-trivial leaves other than connected components of $\partial\mathcal{M}$, which are γ_i. We have done this for $\mathcal{M} = M$ ($c = 0$, $\partial\mathcal{M} = \emptyset$) in Theorem 4.

As in that theorem, consider a tubular neighborhood Y of a wedge sum $\bigvee_i S_i^1$ of circles that generate a basis of $H_1(\mathcal{M})$, foliate it as shown in Fig. 1, and extend the obtained Morse function f to the rest of $\mathcal{M}$. We need, however, a closer look at $\mathcal{M}' = \overline{\mathcal{M} \setminus Y}$ than (7), since now $\partial\mathcal{M}' = \partial Y \cup \partial\mathcal{M}$.

By construction, $i_* H_1(\partial\mathcal{M}') = H_1(\mathcal{M}')$, where $i : \partial\mathcal{M}' \to \mathcal{M}'$ is the inclusion map. Let us consider the commutative diagram:

$$\begin{array}{ccc} H_{n-1}(\mathcal{M}', \partial\mathcal{M}') & \xrightarrow{\partial} & H_{n-2}(\partial\mathcal{M}') \\ \downarrow & & \downarrow \\ H^1(\mathcal{M}') & \xrightarrow{i^*} & H^1(\partial\mathcal{M}') \end{array}$$

where vertical arrows are Poincaré duality. Since by construction i_* is surjective, we have $\ker i^* = 0$. Thus $\ker \partial = 0$. Consider the long exact sequence of a pair:

$$\to H_{n-1}(\partial\mathcal{M}') \xrightarrow{i_*} H_{n-1}(\mathcal{M}') \xrightarrow{j} H_{n-1}(\mathcal{M}', \partial\mathcal{M}') \xrightarrow{\partial} H_{n-2}(\partial\mathcal{M}') \to$$

Since $\operatorname{im} j = \ker \partial = 0$, we obtain $H_{n-1}(\mathcal{M}') = i_* H_{n-1}(\partial\mathcal{M}')$. Thus leaves of f on $\mathcal{M}$ are homologous to 0 or γ_i.

Again, we may assume that $f|_{\partial\mathcal{M}}$ fits together smoothly, giving on M a Morse form ω with $c(\omega) = c$. Obviously, the corresponding foliation is compactifiable. □

REMARK 9. If $\dim M \geqq 3$, not all combinations of $c(\omega)$ and $m(\omega)$ allowed by Theorem 3 may be possible. Inequality (1) imposes additional restrictions on $m(\omega)$ if $b_1'(M) > \frac{1}{2} b_1(M)$. The latter values are independent: for a torus T^n it holds $b_1'(T^n) = 1$, $b_1(T^n) = n$; for a connected sum $M = \sharp_{i=1}^m (S^{n-1} \times S^1)$ it holds $b_1'(M) = b_1(M) = m$ [1].

EXAMPLE 10. Let $M = S^2 \times S^1$; obviously, $b_1'(M) = b_1(M) = 1$. Though Theorem 3 allows $m(\omega) = 1$, (1) prohibits it.

CONJECTURE 11. *On M there exist Morse forms with all combinations of $c(\omega)$ and $m(\omega)$ that satisfy* (6) *and* (1).

6. Generic forms

REMARK 12. Theorems 3, 4, and 8 hold in the class of generic forms for $\dim M \geqq 3$. Propositions 5 and 6 hold in this class for any M.

Indeed, the corresponding Morse forms or functions constructed in their proofs are generic. However, for M_g^2 the three theorems, as well as Proposition 7, should be modified to hold in the class of generic forms.

For the following fact proved in [10], we give a shorter independent proof.

LEMMA 13 (see [10]). *On M_g^2, if ω is generic and $m(\omega) = 0$ ($\mathcal{F}_\omega$ compactifiable) then $c(\omega) = g$.*

PROOF. Consider the foliation graph Γ. Its cycle rank $m(\Gamma) = N_e - N_v + 1$, where N_e is the number of edges and N_v of vertices [5]. Since ω is generic and $\mathcal{F}_\omega$ compactifiable, vertices of Γ are of indices 1 or 3: $2N_e = n_1 + 3n_3$, where n_i is the number of vertices of index i [5]. So $2m(\Gamma) = n_3 - n_1 + 2$. Obviously, $n_1 = |\Omega_0|$ and $n_3 = |\Omega_1|$, where Ω_0 is the set of centers and Ω_1 of conic singularities. By (4), we have $2c(\omega) = |\Omega_1| - |\Omega_0| + 2$. On the other hand, on M_g^2 it holds $|\Omega_1| - |\Omega_0| = 2g - 2$. We obtain $c(\omega) = g$. □

PROPOSITION 14. *The statement of Theorem* 3 *holds for generic forms except that on M_g^2, $g \geqq 1$, the exact lower boundary in* (6) *is* 1*:*

$$1 \leqq c(\omega) + m(\omega) \leqq b_1'(M_g^2) = g.$$

PROOF. That 0 in (6) is unreachable for a generic form on M_g^2, $g \neq 0$, follows from Lemma 13, which together with Lemma 2 gives

$$c(\omega) + m(\omega) = c(\omega) = b_1'(M_g^2) = g.$$

Existence of all intermediate values in (6) in the class of generic forms follows from Proposition 17 and Theorem 8 (Remark 12). Exactness of the lower bound also independently follows from Proposition 15 and Proposition 6 and that of the upper bound from Proposition 5 (Remark 12). □

PROPOSITION 15. *The statement of Theorem* 4 *holds for generic forms iff* $\dim M \geqq 3$ *or* $M = S^2$.

PROOF. For exclusion of M_g^2, $g \geqq 1$, see Lemma 13. □

REMARK 16. Similarly, the statement of Theorem 8 holds for generic forms except that on M_g^2 the form cannot be chosen with $m(\omega) = 0$ unless $c(\omega) = g$.

PROPOSITION 17. *Let* $c, m \in \mathbb{Z}$. *On* M_g^2 *there exists a generic Morse form* ω *such that* $c(\omega) = c$ *and* $m(\omega) = m$ *iff either* $m > 0$ *and* $1 \leqq c + m \leqq g$ *or* $m = 0$ *and* $c = g$ *(cf. Proposition* 7*).*

PROOF. By Lemma 13 and Proposition 14, we only need to show existence. If $m = 0$, represent M_g^2 as a connected sum of g tori with a compact non-singular foliation. Otherwise, represent it as a connected sum of$c$ tori with a compact, and $m - 1$ with a minimal, non-singular foliation plus an $M_{g-c-m+1}^2$ with a foliation as in Proposition 6 (Remark 12). □

REFERENCES

[1] ARNOUX, P. and LEVITT, G., Sur l'unique ergodicité des 1-formes fermées singulières, *Invent. Math.,* **84** (1986), no. 1, 141–156. *MR* **87g**:58004

[2] FARBER, M., *Topology of closed one-forms,* Math. Surv. and Monographs, AMS, v. 108, 2004. *MR* **2005c**:58023

[3] GELBUKH, I., Presence of minimal components in a Morse form foliation,*Diff. Geom. Appl.,* **22** (2005), no. 2, 189–198. *MR* **2005m**:57040

[4] GELBUKH, I., On the structure of a Morse form foliation, *Czechoslovak Mathematical Journal,* **59** (2009), no. 1, 207–220.

[5] HARARY, F., *Graph theory,* Addison-Wesley Publ. Comp., 1994. *MR* **41**#1566

[6] IMANISHI, H., On codimension one foliations defined by closed one forms with singularities, *J. Math. Kyoto Univ.,* **19** (1979), no. 2, 285–291. *MR* **80k**:57050

[7] KATOK, A., Invariant measures for flows on oriented surfaces,*Sov. Math., Dokl.,* **14** (1973), no. 3, 1104–1108. *MR* **48**#9771

[8] LEVITT, G., 1-formes fermées singulières et groupe fondamental,*Invent. Math.,* **88** (1987), 635–667. *MR* **88d**:58004

[9] LEVITT, G., Groupe fondamental de l'espace des feuilles dans les feuilletages sans holonomie, *J. Diff. Geom.,* **31** (1990), 711–761. *MR* **91d**:57018

[10] MEĽNIKOVA, I., An indicator of the noncompactness of a foliation on M_g^2, *Math. Notes,* **53:3** (1993), 356–358. *MR* **94h**:57044

[11] MEĽNIKOVA, I., A test for non-compactness of the foliation of a Morse form,*Russ. Math. Surveys,* **50:2** (1995) 444–445. *MR* **96f**:57028

[12] MEĽNIKOVA, I., Non-compact leaves of a Morse form foliation, *Math. Notes,* **63:6** (1998), 760–763. *MR* **2000e**:57046