

ON COLLINEAR CLOSED ONE-FORMS

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Abstract

We study one-forms with zero wedge-product, which we call *collinear*, and their foliations. We characterise the set of forms that define a given foliation, with special attention to closed forms and forms with small singular sets. We apply the notion of collinearity to give a criterion for existence of a compact leaf and to study homological properties of compact leaves.

2000 *Mathematics subject classification*. primary 57R30; secondary 58A10.

Keywords and phrases: differential one-form, singular set, foliation, compact leaves, cup-product.

1. Introduction and statement of the main results

We consider a closed oriented n -dimensional manifold M and a smooth one-form α on it with the singular set $\text{Sing } \alpha$. This form defines on $\text{Supp } \alpha = M \setminus \text{Sing } \alpha$ a foliation \mathcal{F}_α , which can be extended to the whole M as a singular foliation $\overline{\mathcal{F}}_\alpha$ (Definition 5.2). We shall study the conditions for other forms β to define the same foliation: $\mathcal{F}_\beta = \mathcal{F}_\alpha$ or $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}_\alpha$.

We call forms α, β *collinear*, denoted by $\alpha \parallel \beta$, if $\alpha \wedge \beta = 0$ [10]. Collinear one-forms appear in many problems of theoretical physics, for example, in general relativity: type I vacuum solutions with aligned Papapetrou fields [6] or triplet ansatz [2, 11]. Ranks (of group of the periods) of collinear Morse forms (closed one-forms with non-degenerate singularities) have been studied in [10].

On the intersection of their supports, collinear forms are proportional (Lemma 3.3). Moreover, on this set they share important properties—in particular, integrability (Proposition 3.6) and, if they are integrable, the foliation. Thus the set of the forms that define the same foliation $\overline{\mathcal{F}}_\alpha$ are those that (obviously) have the corresponding support and are collinear with α (Theorem 3.7).

The relation of collinearity is reflexive and symmetric, but generally not transitive (F 1). For Morse forms, however, collinearity is an equivalence relation [10]; we generalise this fact to one-forms with nowhere dense singular sets (Proposition 3.8). In this case, integrability of the form is a class invariant and the singular foliation uniquely identifies integrable classes (i.e., it is a class invariant different for different classes). Similarly, for one-forms with a common support, collinearity is an equivalence relation

Cite this paper as: *Bulletin of the Australian Mathematical Society*, 84(2):322–336, 2011.
Preprint version. Final version: <http://dx.doi.org/10.1017/S0004972711002310>

(Corollary 3.5), integrability is a class invariant, and the foliation uniquely identifies integrable classes.

Closed forms, i.e., $d\alpha = 0$, are an important class of integrable one-forms. They are especially interesting because they allow for using cohomology techniques. In particular, $\alpha \wedge \beta = 0$ implies $[\alpha] \smile [\beta] = 0$, where $\smile : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$ is the cup-product and $[\alpha]$ is the cohomology class; this is used in our study of foliations defined by closed forms (Theorem 7.3). In addition, closed one-forms define an important class of foliations—foliations without holonomy; moreover, any codimension-one foliation without holonomy is topologically equivalent to a foliation defined by a closed one-form [19]. For closed one-forms, singular foliation $\overline{\mathcal{F}}_\alpha$ is a good approximation of the foliation \mathcal{F}_α (Lemma 5.3 to Lemma 5.6).

For closed collinear one-forms, the intersection of their supports consists of entire leaves of both forms (Corollary 4.3). In particular, no leaf of one form can intersect the boundary of the support of the other form (Proposition 4.2).

Our main result states that for closed one-forms with small enough singular sets, their foliations coincide iff the forms are collinear (Theorem 6.5); this has been known for Morse forms [10]. Note that, unlike Theorem 3.7, coincidence of supports is no longer required but is instead guaranteed by the conditions.

Namely, for closed one-forms with at most $(n - 2)$ -dimensional singular sets, it holds $\mathcal{F}_\alpha = \mathcal{F}_\beta$ iff $\alpha \parallel \beta$ (Theorem 6.5); note that for such forms $\alpha \parallel \beta$ implies $\text{Supp } \alpha = \text{Supp } \beta$. We mean here the topological (covering) dimension (Definition 2.1), since $\text{Sing } \alpha$ is generically not a submanifold.

The condition on $\text{Sing } \alpha$ can be relaxed: for closed one-forms with at most $(n - 1)$ -dimensional, i.e., nowhere dense (Lemma 2.3), singular sets, it holds $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$ iff $\alpha \parallel \beta$ (Theorem 6.5). Further generalisations include: for (not necessarily closed) one-forms with $\dim \text{Sing } \alpha \leq 0$, $\alpha \parallel \beta$ implies $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$. (Proposition 6.2); for closed one-forms (irrespective of their singular sets) $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$ implies $\alpha \parallel \beta$ (Proposition 6.4).

The notion of collinearity is useful for the study of foliations defined by closed one-forms, as can be illustrated by the following examples.

Farber *et al.* [4, 5] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form, in terms of the cup-product. We generalise this condition in terms of collinearity to an arbitrary closed one-form and improve it to a criterion (Theorem 7.2).

In [10], it was shown that the topology of a foliation defined by a Morse form of the maximum possible rank for the given M , $\text{rk } \alpha = b_1(M)$ (the first Betti number) is connected with the structure of the cup-product: $c(\alpha) \leq \text{rk ker } \smile$, where $c(\alpha)$ is the maximum number of homologically independent compact leaves of \mathcal{F}_α , an important value in the theory of foliations defined by Morse forms [8, 9]. We use the notion of collinearity to generalise this fact to arbitrary closed one-forms (Theorem 7.3).

The paper is organised as follows. In Section 2 we give various conditions for $\text{Sing } \alpha$ to be small, in terms of covering dimension. In Section 3 we introduce collinear forms and consider some their properties. In Section 4 we study the supports of closed

collinear one-forms. In Section 5 we define the notion of singular foliation and show that it is well-behaved for closed forms. In Section 6 we prove our main result on the characterisation of the set of closed forms with small supports that define the same foliation, as an equivalence class of collinear forms. Finally, in Section 7 we use the notion of collinearity to study homological properties of compact leaves of a foliation defined by a closed one-form.

2. Singular set

We consider smooth one-forms on a closed oriented n -dimensional manifold M . One-form α is a smooth section of the cotangent bundle of M , i.e., it assigns to each point $x \in M$ a linear functional $\alpha_x : T_x M \rightarrow \mathbb{R}$. Denote

$$\text{Sing } \alpha = \{x \in M \mid \alpha_x = 0\}.$$

Obviously, $\text{Sing } \alpha \subseteq M$ is closed and the support $\text{Supp } \alpha = M \setminus \text{Sing } \alpha$ is open. Generically, $\text{Sing } \alpha$ is not a submanifold.

To characterise a subset of M that is not necessarily a submanifold, we use the topological (covering) dimension:

D 2.1 ([12]). The *covering dimension* $\dim X$ of a topological space X is the minimum value n such that every open cover of X has an open refinement in which no point is included in more than $n + 1$ elements.

For instance, $\dim \emptyset = -1$; the covering dimension of a simplicial complex is the maximum dimension of its simplices; $\dim(S^2 \vee S^1) = 2$. If $Y \subseteq X$, then $\dim Y \leq \dim X$. Covering dimension of a manifold coincides with its conventional dimension [12, Corollary 1 of Theorem IV.3]. In particular:

R 2.2. If $X \subseteq M$ and $X \supseteq i\mathbb{R}^k$ (i is an inclusion), then $\dim X \geq k$.

For $X \subseteq M$, $\dim X \leq n - 1$ iff interior $\text{int}(X) = \emptyset$ [12, Theorem IV.3]; in particular, the covering dimension of a leaf of a foliation is $n - 1$. If $\dim X \leq n - 2$ then X does not locally divide M [12, Theorem IV.4].

Thus $\dim \text{Sing } \alpha \leq n = \dim M$; obviously, $\partial \text{Supp } \alpha \subseteq \text{Sing } \alpha = M \setminus \text{Supp } \alpha$.

L 2.3. For a one-form α , the following conditions are equivalent:

- (i) $\text{Sing } \alpha$ is nowhere dense in M ;
- (ii) $\text{Supp } \alpha$ is dense in M ;
- (iii) $\partial \text{Supp } \alpha = \text{Sing } \alpha$;
- (iv) $\text{int}(\text{Sing } \alpha) = \emptyset$;
- (v) $\dim \text{Sing } \alpha \leq n - 1$.

P . (ii) \Leftrightarrow (iii): $Y \subseteq X$ is open and dense iff $\partial Y = Y^C$, the complement.
 (iii) \Leftrightarrow (iv): $\text{int}(Y) = \emptyset$ iff $Y = \partial(Y^C)$. (iv) \Leftrightarrow (v): $\dim Y < \dim X$ iff $\text{int}(Y) = \emptyset$ [12, Theorem IV.3]. (iv) \Leftrightarrow (i): $\text{Sing } \alpha$ is closed. □

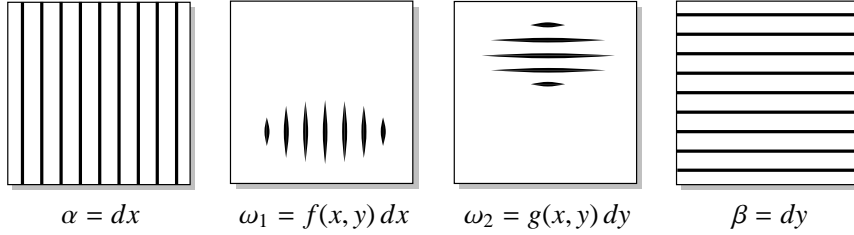
3. Collinear one-forms

D 3.1. Two differential forms α, β are called *collinear* if $\alpha \wedge \beta = 0$; we denote this as $\alpha \parallel \beta$.

The term is motivated by Lemma 3.3 below.

R 3.2. The set of $x \in M$ at which $\alpha \parallel \beta$ is closed.

Denote $S(\alpha, \beta) = \text{Supp } \alpha \cap \text{Supp } \beta$; it is open. The collinearity relation is reflexive and symmetric but not necessarily transitive: obviously, $S(\alpha, \beta) = \emptyset$ implies $\alpha \parallel \beta$, so any two forms α, β are connected via two forms ω_j such that $\alpha \parallel \omega_1, \omega_1 \parallel \omega_2$, and $\omega_2 \parallel \beta$ even if $\alpha \wedge \beta \neq 0$; see F 1; cf. [18]. Therefore collinearity is generally not an equivalence relation. In the sequel we, however, shall consider various conditions under which collinearity is an equivalence.



F 1. Chain of collinear forms on a 2-torus. Note that $S(\omega_1, \omega_2) = \emptyset$; f and g are bump functions.

L 3.3. Let α, β be one-forms. The following conditions are equivalent:

- (i) $\alpha \parallel \beta$,
- (ii) $\alpha = f(x)\beta$ on $\text{Supp } \beta$,
- (iii) $\beta = g(x)\alpha$ on $\text{Supp } \alpha$.

On $S(\alpha, \beta)$, the functions $f(x), g(x)$ are smooth and non-vanishing, with $f(x)g(x) = 1$.

P . (ii) or (iii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii) and similarly (iii): Consider on $\text{Supp } \beta$ a smooth vector field ξ with non-vanishing $\beta(\xi_x)$. Choose

$$f(x) = \frac{\alpha(\xi_x)}{\beta(\xi_x)},$$

which, by collinearity, does not depend on the choice of ξ : indeed, for any other such field ξ' collinearity gives

$$(\alpha \wedge \beta)(\xi_x, \xi'_x) = \alpha(\xi_x)\beta(\xi'_x) - \alpha(\xi'_x)\beta(\xi_x) = 0.$$

Thus $\alpha = f(x)\beta$ on $\text{Supp } \beta$; in addition, $f(x) \neq 0$ on $S(\alpha, \beta)$. □

So on $S(\alpha, \beta)$ collinear forms are proportional; thus the term. However, this does not necessarily hold on the whole M : for instance, any two forms with non-intersecting supports are collinear, but not proportional.

C 3.4. On $\text{Supp}\beta$, $\alpha \parallel \beta$ and $\beta \parallel \gamma$ imply $\alpha \parallel \gamma$.

C 3.5. Consider a set A of one-forms on M such that $S(\alpha, \beta) = S \subseteq M$ is the same for any $\alpha, \beta \in A$; for instance, $\text{Supp}\alpha = S$ for any $\alpha \in A$. Then the collinearity is an equivalence relation on A .

On $S(\alpha, \beta)$, collinear forms share important properties. Consider an integrable form α ; it defines a foliation \mathcal{F}_α on $\text{Supp}\alpha$.

P 3.6. Let α be an integrable one-form. Then for a one-form β it holds $\beta \parallel \alpha$ iff on $S = S(\alpha, \beta)$ it is also integrable and $\mathcal{F}_\beta|_S = \mathcal{F}_\alpha|_S$.

By $\mathcal{F}|_S$ we understand a foliation of S whose leaves are path-connected components of $\gamma \cap S$, $\gamma \in \mathcal{F}$.

P . Let $\alpha \parallel \beta$. By the Frobenius theorem, α is integrable iff $\alpha \wedge d\alpha = 0$. By Lemma 3.3, on S it holds $\beta = f(x)\alpha$, so $\beta \wedge d\beta = 0$ and thus β is also integrable. The foliations coincide since $\{\alpha = 0\} \equiv \{f\alpha = 0\}$.

Assume now $\mathcal{F}_\alpha = \mathcal{F}_\beta$ on $S(\alpha, \beta)$. Outside $S(\alpha, \beta)$ the forms are obviously collinear. Consider $x \in S(\alpha, \beta)$ and $\xi_x, \eta_x \in T_xM = \langle n_x, T_x\gamma \rangle$, where $\gamma \in \mathcal{F}_\alpha$ is the leaf that contains x and $n_x \in T_xM \setminus T_x\gamma$; $\xi_x = an_x + \xi'_x$, $\eta_x = bn_x + \eta'_x$ for some $a, b \in \mathbb{R}$ and $\xi'_x, \eta'_x \in T_x\gamma$. Since $\alpha|_{T_x\gamma} = \beta|_{T_x\gamma} = 0$, we have

$$(\alpha \wedge \beta)(\xi_x, \eta_x) = \alpha(an_x)\beta(bn_x) - \alpha(bn_x)\beta(an_x) = 0,$$

i.e. $\alpha \parallel \beta$. □

Therefore, collinear forms are either both integrable or both non-integrable on $S(\alpha, \beta)$, i.e., integrability on S is a class invariant under the equivalence relation from Corollary 3.5. Integrable one-forms α, β are collinear iff $\mathcal{F}_\alpha = \mathcal{F}_\beta$ on $S(\alpha, \beta)$, i.e., $\mathcal{F}|_S$ uniquely identifies an integrable class under this equivalence relation.

We have obtained a characterisation of the set of forms defining a given foliation \mathcal{F} as the equivalence class, under collinearity relation, of forms with the corresponding support. In particular:

T 3.7. Given a foliation $\mathcal{F} = \mathcal{F}_\alpha$ of a one-form α , the forms defining \mathcal{F} are one-forms collinear with α and having the same support.

The latter condition automatically holds—and thus can be omitted from Theorem 3.7—in the class of closed forms with small enough singular sets; this is formulated below as part of Theorem 6.5.

Forms with small singular sets give another important class of forms on which collinearity is an equivalence relation:

P 3.8. On the set of one-forms with nowhere dense singular sets (cf. Lemma 2.3), collinearity is an equivalence relation.

P . It is sufficient to prove transitivity. Suppose $\alpha \parallel \beta$ and $\beta \parallel \gamma$, but $\alpha \not\parallel \gamma$ at $x \in M$ and thus, by Remark 3.2, on some open $U \neq \emptyset$. By Corollary 3.4, $U \subseteq \text{Sing}\beta$; a contradiction. \square

In this case, integrability on the whole M is a class invariant (by Proposition 3.6 and the fact that $\alpha \wedge d\alpha$ is continuous) but the foliation is not, as can be seen in Figure 3. For closed such forms, however, the singular foliation introduced below in Definition 5.2 is a class invariant uniquely identifying an equivalence class. This is also formulated below as part of the same Theorem 6.5.

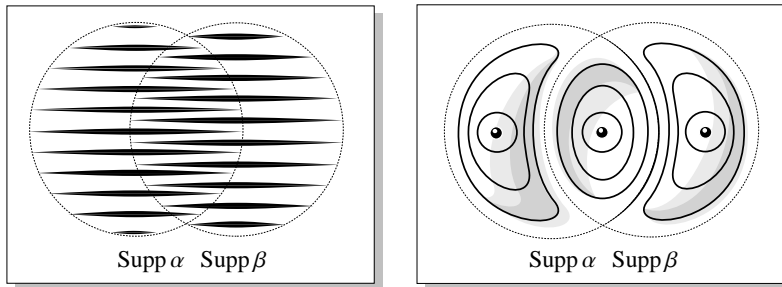
4. Supports of closed collinear one-forms

In the sequel we consider an important class of integrable forms: closed one-forms. Properties of the supports of closed collinear one-forms prove to be connected with their foliations.

L 4.1. Let α, β be closed one-forms and $\alpha = f(x)\beta$ on $\text{Supp}\beta$. Then $f(x)$ is constant on leaves of \mathcal{F}_β .

This follows by direct calculation from the equation $d\alpha = df \wedge \beta = 0$.

By Proposition 3.6, leaves of collinear integrable one-forms α, β coincide on $S = S(\alpha, \beta)$, i.e., $\mathcal{F}_\alpha|_S = \mathcal{F}_\beta|_S$. The following proposition shows that if both forms are closed, then leaves of one form cannot intersect the boundary of the support of the other form; see Figure 2.

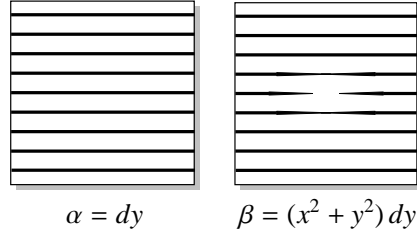


F 2. Left: this is possible only for non-closed collinear forms; right: foliations of closed collinear forms.

P 4.2. Let α, β be closed collinear one-forms and $\gamma_\alpha \in \mathcal{F}_\alpha$. Then $\gamma_\alpha \cap \text{Supp}\beta \neq \emptyset$ implies $\gamma_\alpha \subset \text{Supp}\beta$ and moreover $\gamma_\alpha \in \mathcal{F}_\beta$ (and similarly for $\text{Sing}\beta$).

P . By Lemma 3.3, on $\text{Supp } \alpha$ it holds $\beta = f(x)\alpha$. Let $\gamma_\alpha \cap \text{Supp } \beta \neq \emptyset$. If $\gamma_\alpha \not\subset \text{Supp } \beta$, then there exists $x_0 \in \gamma_\alpha \cap \text{Sing } \beta$, so $f(x_0) = 0$. By Lemma 4.1, we have $f(x) = 0$ on γ_α , i.e. for all $x \in \gamma_\alpha$ it holds $\beta_x = 0$, thus $\gamma_\alpha \subset \text{Sing } \beta$; a contradiction. By Proposition 3.6, $\gamma_\alpha \in \mathcal{F}_\beta$. \square

The condition for the forms to be closed is important. Indeed, on a torus $M = T^2$ consider $\alpha = dy$, $\beta = (x^2 + y^2)dy$ locally (while far from the singularity, assume this coefficient to be 1); see F 3. Then $\text{Supp } \alpha = M$, $\text{Supp } \beta = M \setminus \{0\}$; $\gamma = \{y = 0\} \in \mathcal{F}_\alpha$ intersects both $\text{Supp } \beta$ and $\text{Sing } \beta = \{0\}$.



F 3. One-forms on a 2-torus (far from the singularity, the coefficient is assumed to be 1).

C 4.3. Let α, β be closed collinear one-forms; $\gamma_\alpha \in \mathcal{F}_\alpha$, $\gamma_\beta \in \mathcal{F}_\beta$. Then $\gamma_\alpha \cap \gamma_\beta \neq \emptyset$ implies $\gamma_\alpha = \gamma_\beta$. In particular,

$$S(\alpha, \beta) = \bigcup_{\gamma \in \mathcal{F}_\alpha \cap \mathcal{F}_\beta} \gamma.$$

Indeed, by Proposition 4.2, $\gamma_\alpha, \gamma_\beta \subset S(\alpha, \beta)$, then by Proposition 3.6, $\gamma_\alpha = \gamma_\beta$.

In other words, leaves of α and β coincide if intersect, and the intersection of their supports consists of entire leaves of both forms.

5. Singular foliation of a closed form

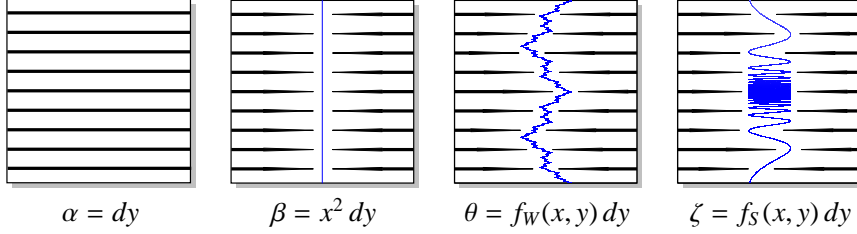
Let α be an integrable one-form. Its foliation is defined only on $\text{Supp } \alpha$, namely:

R 5.1. A foliation \mathcal{F}_α is a decomposition of $\text{Supp } \alpha$ into leaves: two points $p, q \in \text{Supp } \alpha$ belong to the same leaf γ iff there exists a smooth path $s: [0, 1] \rightarrow \text{Supp } \alpha$, $s(0) = p$, $s(1) = q$, such that $\alpha(\dot{s}(t)) \equiv 0$, $t \in [0, 1]$.

This notion can be extended to the whole M to define a so-called singular foliation. In [4], singular foliation for Morse forms (their singular sets are finite) is defined as in Remark 5.1 by substituting M for $\text{Supp } \alpha$. For arbitrary integrable forms we, however, prefer to relax the smoothness condition:

D 5.2. A singular foliation $\overline{\mathcal{F}}_\alpha$ is a decomposition of M into leaves: two points $p, q \in M$ belong to the same leaf $\tilde{\gamma}$ iff there exists a continuous path $s: [0, 1] \rightarrow M$, $s(0) = p$, $s(1) = q$, such that on $s^{-1}(\text{Supp } \alpha) \setminus \{0, 1\}$ it is differentiable and $\alpha(\dot{s}(t)) \equiv 0$.

In F 4, $\overline{\mathcal{F}}_\alpha = \mathcal{F}_\alpha$; $\overline{\mathcal{F}}_\beta = \{\tilde{\gamma}\}$, $\tilde{\gamma} = M$; and $\mathcal{F}_\zeta = \{\tilde{\gamma}_1, \tilde{\gamma}_2\}$, $\tilde{\gamma}_1 = S^1$. With the relaxed smoothness condition as in Definition 5.2, we have $\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_\beta$, while if we required the path to be smooth on M , we would have $\overline{\mathcal{F}}_\theta = \overline{\mathcal{F}}_\alpha$, which is, perhaps, a matter of personal taste.



F 4. Smooth integrable one-forms on a 2-torus $M = T^2$ (the coefficients are assumed to be 1 far from the singular set). $\text{Sing } \theta$ is the graph of the Weierstrass function, which is continuous but nowhere differentiable. As a closed subset of M , it is the set of zeros of a smooth function f_W [13, Theorem 1.5 ff.]. Note that this path-connected set does not contain any non-trivial differentiable path $s: [0, 1] \rightarrow M$. The function f_S is obtained similarly from the closed topologist's sine curve; it is connected but not path-connected.

For any $\gamma \in \mathcal{F}_\alpha$, if $\gamma \cap \tilde{\gamma} \neq \emptyset$, then $\gamma \subseteq \tilde{\gamma}$, i.e., a singular leaf of $\overline{\mathcal{F}}_\alpha$ consists of entire leaves of \mathcal{F}_α and path-connected components of $\text{Sing } \alpha$.

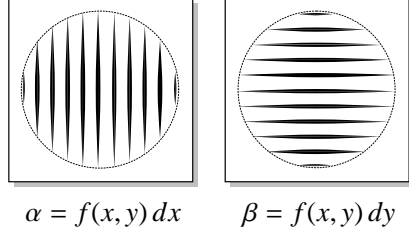
Unlike leaves γ of a usual foliation \mathcal{F} , which have the same dimension $\dim \gamma = n - 1$, leaves $\tilde{\gamma}$ of a singular foliation $\overline{\mathcal{F}}$ can have different dimensions $0 \leq \dim \tilde{\gamma} \leq n$. For instance, a center singularity is a 0-dimensional singular leaf; if $\text{int}(\text{Sing } \alpha) \neq \emptyset$, then there exists $\tilde{\gamma} \in \overline{\mathcal{F}}_\alpha$ with $\dim \tilde{\gamma} = n$ (cf. Lemma 2.3). By Remark 5.1, if $\dim \tilde{\gamma} \neq n - 1$, then $\tilde{\gamma} \cap \text{Sing } \alpha \neq \emptyset$; in particular, $\dim \tilde{\gamma} \leq n - 2$ implies that $\tilde{\gamma} \subseteq \text{Sing } \alpha$ is a path-connected component of $\text{Sing } \alpha$.

The notion of singular foliation as defined by Definition 5.2, while well-defined, is rather counter-intuitive for non-closed forms. As seen in F 4, for a leaf $\tilde{\gamma} \in \overline{\mathcal{F}}_\alpha$ it is possible that $\dim(\tilde{\gamma} \cap \text{Supp } \alpha) = n$ or even $\tilde{\gamma} = M$ with $\text{Supp } \alpha \neq \emptyset$. Thus for a curve $s(t) \subseteq \tilde{\gamma}$ it is possible that $\alpha(\dot{s}(t)) \neq 0$. Two non-collinear forms with $\text{Supp } \alpha = \text{Supp } \beta$ can have $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$, as in F 5.

For closed forms, however, the notion of $\overline{\mathcal{F}}_\alpha$ is quite intuitive and is a good approximation of \mathcal{F}_α , as we show in the rest of this section. The form β in F 4 shows that closedness is important in Lemma 5.3 to Lemma 5.6.

L 5.3. Let α be a closed one-form, $\tilde{\gamma} \in \overline{\mathcal{F}}_\alpha$, and $s(t) \subseteq \tilde{\gamma}$ a differentiable curve. Then $\alpha(\dot{s}(t)) \equiv 0$.

P . Consider $F(x) = \int_0^x \alpha(\dot{s}(t)) dt$. Since $s \subseteq \tilde{\gamma}$, for any $x \in [0, 1]$ there exists a path $u_x: [0, 1] \rightarrow M$ connecting $s(x)$ and $s(0)$, such that $\int_0^1 \alpha(\dot{u}_x(t)) dt = 0$. Thus $F(x) \in P$, where $P = \langle \int_z \alpha \mid z \in H_1(M)_{\mathbb{Z}} \rangle$ is the group of periods. Since P is finitely



F 5. Non-closed one-forms on a 2-torus $M = T^2$; f is a bump function. $\text{Supp } \alpha = \text{Supp } \beta$ and $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta = \{\tilde{\gamma}\}$, $\tilde{\gamma} = M$, but $\alpha \not\parallel \beta$.

generated and thus at most countable while F is continuous, we have $F \equiv \text{const}$ and thus $\alpha(\dot{s}(t)) \equiv 0$. □

C 5.4. Let α be a closed one-form and $\tilde{\gamma} \in \overline{\mathcal{F}}_\alpha$. Then $\text{int}(\tilde{\gamma}) \subseteq \text{Sing } \alpha$.

C 5.5. For a closed one-form, $\text{Sing } \alpha$ is nowhere dense iff $\dim \tilde{\gamma} \leq n - 1$ for any $\tilde{\gamma} \in \overline{\mathcal{F}}_\alpha$.

L 5.6. For a closed form, $\overline{\mathcal{F}}_\alpha|_{\text{Supp } \alpha} = \mathcal{F}_\alpha$.

By $\overline{\mathcal{F}}|_S$ we understand, again, a decomposition of S into path-connected components of $\tilde{\gamma} \cap S$, $\tilde{\gamma} \in \overline{\mathcal{F}}$.

P . Consider a path-connected component $X \subseteq \tilde{\gamma} \cap \text{Supp } \alpha$ and $x \in X$. Then $x \in \gamma$ for some $\gamma \in \mathcal{F}_\alpha$, so $\gamma \subseteq X$. For any $x' \in X \setminus x$ there exists a curve $s(t) \subseteq X \subseteq \tilde{\gamma}$, $s(0) = x$, $s(1) = x'$. By Lemma 5.3, $\alpha(\dot{s}(t)) = 0$; by Remark 5.1, $x' \in \gamma$. Thus $X = \gamma$. □

6. Collinear forms with small singular sets

We shall show that for closed one-forms with small enough singular sets, collinearity implies coincidence of their singular foliations or, in case of even smaller singular sets, coincidence of their (conventional) foliations and, in particular, their supports.

Recall that $\dim X$ stands for the covering dimension. On the set of one-forms with $\dim \text{Sing } \alpha \leq n - 1$, collinearity is an equivalence relation (Proposition 3.8).

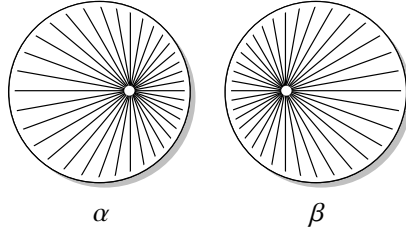
L 6.1. Let $\alpha \parallel \beta$ be integrable one-forms, with $\dim \text{Sing } \alpha \leq 0$. Then for any $\tilde{\gamma}_\alpha \in \overline{\mathcal{F}}_\alpha$ there exists $\tilde{\gamma}_\beta \in \overline{\mathcal{F}}_\beta$ such that $\tilde{\gamma}_\alpha \subseteq \tilde{\gamma}_\beta$.

P . Let $p, q \in \tilde{\gamma}_\alpha$, i.e., there exists $s: [0, 1] \rightarrow M$, $s(0) = p$, $s(1) = q$ such that $\alpha(\dot{s}(t)) = 0$. Let us show that $\beta(\dot{s}(t)) = 0$, then $\tilde{\gamma}_\alpha \subseteq \tilde{\gamma}_\beta$.

Suppose $\beta(\dot{s}(t)) \neq 0$ for some $t \in [0, 1]$. Since $\dim \alpha \leq 0$, by Remark 2.2 for any small neighbourhood $U(t)$ there exists $t' \in U$ such that $s(t') \in \text{Supp } \alpha$. By Lemma 3.3, in a small neighbourhood of $s(t')$ we have $\beta = f(x)\alpha$, thus $\beta(\dot{s}(t')) = 0$. We obtain $\beta(\dot{s}(t)) = 0$; a contradiction. □

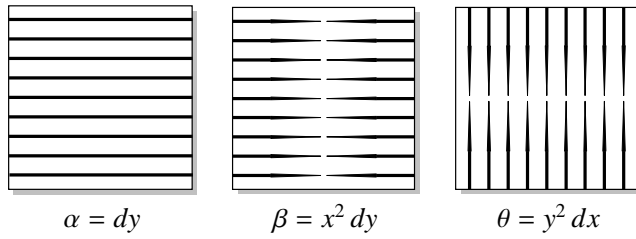
P 6.2. Let $\alpha \parallel \beta$ be integrable one-forms with $\dim \text{Sing } \alpha \leq 0$ and $\dim \text{Sing } \beta \leq 0$. Then $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$.

The converse is not true: F 6 shows two “meridional” forms $\alpha \parallel \beta$ on S^2 with different “poles”; however, $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$.



F 6. Non-closed one-forms $\alpha \parallel \beta$ on $M = S^2$, each one with two singularities; $\dim \text{Sing } \alpha = \dim \text{Sing } \beta = 0$, $\text{Supp } \alpha \neq \text{Supp } \beta$, but $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta = \{\gamma\}$, $\tilde{\gamma} = M$.

The condition $\dim \text{Sing } \alpha \leq 0$ in Lemma 6.1 and Proposition 6.2 cannot be relaxed to a higher value; cf. the forms α and β in F 7. Note that an example analogous to β can be constructed in any dimension n , with $\dim \text{Sing } \beta = 1$.



F 7. One-forms on a 2-torus $M = T^2$ (far from the singular set, the coefficients are equal to 1). $\text{Sing } \beta = S^1$; $\text{Sing } \theta = S^1$; $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}_\theta = \{M\}$.

L 6.3. Let $\alpha \parallel \beta$ be closed one-forms and $\dim \text{Sing } \alpha \leq n - 2$. Then $\text{Sing } \alpha \subseteq \text{Sing } \beta$.

P . Suppose there exists $x \in \text{Sing } \alpha \cup \text{Supp } \beta$. Consider $\gamma_\beta \in \overline{\mathcal{F}}_\beta$ such that $x \in \gamma_\beta$. By Proposition 4.2, $\gamma_\beta \subset \text{Sing } \alpha$. Remark 2.2 gives a contradiction. \square

In particular, the supports of collinear Morse forms (closed one-forms with non-degenerated singularities) coincide [10]. The condition for the forms to be closed is important; cf. F 3.

Recall that by Proposition 3.6, $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$ implies $\alpha \parallel \beta$.

P 6.4. Let α be a closed one-form. Then $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}_\alpha$ implies $\beta \parallel \alpha$.

P . Let $x \in S = S(\alpha, \beta)$. Consider a curve $s(t) \subseteq \gamma_\beta \in \mathcal{F}_\beta$, $x \in s(t)$. Then $s(t) \subseteq \tilde{\gamma}_\beta = \tilde{\gamma}_\alpha \in \overline{\mathcal{F}}_\alpha$ and by Lemma 5.3 we have $\alpha(\dot{s}(t)) = 0$. Thus $\{\beta_x = 0\} \subseteq \{\alpha_x = 0\}$, then $\{\beta_x = 0\} = \{\alpha_x = 0\}$ and $\beta_x \parallel \alpha_x$. \square

Now we are ready to formulate our main result:

T 6.5. Let α, β be closed forms with nowhere dense singular sets (cf. Lemma 2.3). Then $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$ iff $\alpha \parallel \beta$.

If, moreover, $\dim \text{Sing } \alpha \leq n - 2$ and $\dim \text{Sing } \beta \leq n - 2$, then $\mathcal{F}_\alpha = \mathcal{F}_\beta$ iff $\alpha \parallel \beta$ (in particular, $\alpha \parallel \beta$ implies $\text{Sing } \alpha = \text{Sing } \beta$).

P . Let $\alpha \parallel \beta$. Consider a leaf $\tilde{\gamma}_\alpha \in \overline{\mathcal{F}}_\alpha$. Let $s: [0, 1] \rightarrow \tilde{\gamma}_\alpha$ be a map from Definition 5.2, so that $\alpha(\dot{s}(t)) = 0$ for all $t \in [0, 1]$. Then also $\beta(\dot{s}(t)) = 0$.

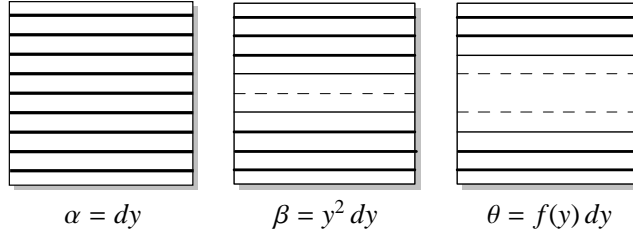
Indeed, suppose at $s(t) \in \text{Supp } \beta$, and thus in a small neighbourhood $U = U(s(t)) \subset \text{Supp } \beta$, the curve s is transversal to the leaves of \mathcal{F}_β . By Lemma 3.3, on $\text{Supp } \beta$ it holds $\alpha = f(x)\beta$, so we have $f(x) = 0$ for $s \cap U$. Then Lemma 4.1 gives $f(x) \equiv 0$ in U , so $U \subseteq \text{Sing } \alpha$; a contradiction. Thus $\beta(\dot{s}(t)) = 0$.

We obtain $\tilde{\gamma}_\alpha \subseteq \tilde{\gamma}_\beta \in \overline{\mathcal{F}}_\beta$ and vice versa; i.e., $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$. The converse follows from Proposition 6.4. Finally, $\mathcal{F}_\alpha = \mathcal{F}_\beta$ follows from Lemma 6.3 and Proposition 3.6. \square

Note that $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$ here is not trivially true, as in F 7, since by Corollary 5.4 in the conditions of the theorem it holds $\dim \tilde{\gamma} \leq n - 1$.

The condition for the forms to be closed is important: in F 4, $\alpha \parallel \beta$ but $\overline{\mathcal{F}}_\alpha \neq \overline{\mathcal{F}}_\beta$; in F 7, $\overline{\mathcal{F}}_\beta = \overline{\mathcal{F}}_\theta$ but $\beta \not\parallel \theta$. This condition is also important for the second part of the theorem: in F 3, $\alpha \parallel \beta$ and $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$, but $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ because $\text{Sing } \alpha \neq \text{Sing } \beta$.

The condition $\dim \text{Sing } \beta \leq n - 2$ in the second part of Theorem 6.5 is important: in F 8, $\alpha \parallel \beta$ are closed forms and therefore $\overline{\mathcal{F}}_\alpha = \overline{\mathcal{F}}_\beta$, but $\mathcal{F}_\alpha \neq \mathcal{F}_\beta$ because $\text{Sing } \alpha \neq \text{Sing } \beta$.



F 8. Closed collinear one-forms on a 2-torus: $\text{Sing } \alpha = \emptyset$, $\text{Sing } \beta = S^1$, $\text{Sing } \theta = I \times S^1$.

Finally, the condition for the singular sets to be nowhere dense is important: in F 8, where $\text{Sing } \theta$ is a band, $\overline{\mathcal{F}}_\alpha \neq \overline{\mathcal{F}}_\theta$. Another example is F 2, right.

Theorem 6.5 describes the set of closed one-forms with small enough singular sets that define the same foliation \mathcal{F} , eliminating the requirement of coincidence of singular sets in Theorem 3.7:

C 6.6. *Let A be the class of closed one-forms with $\dim \text{Sing } \alpha \leq n-2$. Given a foliation $\mathcal{F} = \mathcal{F}_\alpha$ of a form $\alpha \in A$, the forms from A defining \mathcal{F} are those collinear with α .*

Similarly, for the class A of closed one-forms with nowhere dense singular set, the forms from A defining $\overline{\mathcal{F}} = \overline{\mathcal{F}_\alpha}$ are those collinear with α .

Note that the only property of covering dimension used in our results regarding $\dim \text{Sing } \alpha \leq n-2$ is Remark 2.2: if $\dim X \leq k$ then X does not contain \mathbb{R}^k . This weaker condition can be used in Lemma 6.3, Theorem 6.5, and Corollary 6.6: instead of $\dim \text{Sing } \alpha \leq n-2$ it is enough to require for $\text{Sing } \alpha$ not to contain a smooth inclusion of \mathbb{R}^{n-1} . This generalises those results to forms with $\dim \text{Sing } \alpha = n-1$ whose singular set is just not smooth enough to contain \mathbb{R}^{n-1} , such as θ in F 4.

7. Applications of collinearity: foliations of closed one-forms

The notion of collinearity is a useful tool for studying the topology of foliations of closed one-forms.

Since a foliation defined by a closed one-form has no holonomy, by the Reeb local stability theorem each its compact leaf has a neighbourhood consisting of compact leaves. It can be explicitly constructed:

L 7.1. *Let α be a closed one-form. Then any compact leaf $\gamma \in \mathcal{F}_\alpha$ has a cylindrical neighbourhood consisting of compact leaves.*

P . Let U be a neighbourhood of γ where the form is exact: $\alpha = df$; assume $f|_\gamma = 0$. For some $\varepsilon > 0$ consider a neighbourhood $U_\varepsilon = \{x \in U \mid |f(x)| < \varepsilon\}$. Since $\text{Supp } \alpha$ is open, we can choose ε such that $U_\varepsilon \cap \text{Sing } \alpha = \emptyset$. Obviously, $\gamma \subset U_\varepsilon$. By construction,

$$U_\varepsilon = \bigcup_{|y| < \varepsilon} \gamma_y \simeq \gamma \times (-\varepsilon, \varepsilon),$$

where $\gamma_y = f^{-1}(y)$ are compact leaves of \mathcal{F}_α . □

By choosing a suitable form $\beta \parallel \alpha$ one can characterise the topology of the foliation \mathcal{F}_α . For example, if there exists an exact form $df \parallel \alpha$ with $\text{Supp } df = \text{Supp } \alpha$, then the foliation \mathcal{F}_α is compactifiable, i.e., for any leaf $\gamma \in \mathcal{F}_\alpha$ the set $\gamma \cup \text{Sing } \alpha$ is compact. Indeed, the foliation $\mathcal{F}_{df} = \mathcal{F}_\alpha$ is defined by levels of the function $f(x)$, which are compact.

Farber *et al.* [4, 5] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form in terms of the cup-product: if \mathcal{F}_α has a compact leaf γ , $[\gamma] \neq 0$, then there exists a closed one-form β , $0 \neq [\beta] \in H^1(M, \mathbb{Z})$, such that $[\alpha] \smile [\beta] = 0$. This is not a criterion; moreover, no

sufficient condition for existing of a compact leaf can be given in purely cohomologous terms since in any cohomology class $[\alpha]$, $\text{rk } \alpha > 1$, there exists a form with minimal foliation [1]; $\text{rk } \alpha$ is the rank of its group of periods (integrals over one-cycles).

We generalise this condition in terms of collinearity to an arbitrary closed one-form and improve it to a criterion:

T 7.2. *Let α be a closed one-form. The following conditions are equivalent:*

- (i) \mathcal{F}_α has a compact leaf γ ;
- (ii) There exists a smooth function $f(x)$ such that
 - (a) $df \parallel \alpha$,
 - (b) $\text{Supp } \alpha \cap \text{Supp } df \neq \emptyset$;
- (iii) There exists a smooth closed one-form β such that
 - (a) $\beta \parallel \alpha$,
 - (b) $\text{Supp } \alpha \cap \text{Supp } \beta \neq \emptyset$,
 - (c) $[\beta] \in H^1(M, \mathbb{Z})$.

Given a specific γ , f , or β , the other two can be chosen such that $f|_\gamma = \text{const}$ and $[\beta] = D[\gamma]$, where $D: H_{n-1}(M) \rightarrow H^1(M, \mathbb{Z})$ is the Poincaré duality map.

P . (i) \Rightarrow (ii): Consider the function from Lemma 7.1 defined on a cylindrical neighbourhood $\gamma \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$ consisting of compact diffeomorphic leaves, and extend it to a smooth function f constant on leaves, such that $f \equiv 0$ outside $\gamma \times (-\varepsilon, \varepsilon)$. Then $\text{Supp } \alpha \cap \text{Supp } df \neq \emptyset$. By Proposition 3.6, $df \parallel \alpha$.

(ii) \Rightarrow (iii): Consider $\beta = df$; $[\beta] = 0$.

(iii) \Rightarrow (i): Since $[\beta] \in H^1(M, \mathbb{Z})$, the form β defines a map $F: M \rightarrow S^1$,

$$F(x) = e^{2\pi i \int_{x_0}^x \beta}.$$

Obviously, F is constant on leaves of \mathcal{F}_β and the critical set of F coincides with $\text{Sing } \beta$. Consider $F_S = F|_S: S \rightarrow S^1$, where $S = \text{Supp } \alpha \cap \text{Supp } \beta$.

By Sard's theorem, the map F_S has a regular value $y \in S^1$, and by the implicit function theorem, $F_S^{-1}(y)$ is a closed codimension-one submanifold. By Corollary 4.3, the set S consists of entire leaves of both \mathcal{F}_α and \mathcal{F}_β , so a connected component of $F_S^{-1}(y)$ is a compact leaf $\gamma \in \mathcal{F}_\beta \cap \mathcal{F}_\alpha$. Thus (iii) \Rightarrow (i).

Now, given a specific γ , the function f constructed above satisfies $f|_\gamma = \text{const}$. Let us construct β such that $[\beta] = D[\gamma]$.

Construct the function f as above; denote $U = \gamma \times (-\varepsilon, \varepsilon)$. Consider a curve $s(t) \subset U$, $t \in (-\varepsilon, \varepsilon)$, transversal to leaves. Obviously, the form $f(x)\alpha$ is closed; denote $A = \int_{-\varepsilon}^{\varepsilon} f(s(t))\alpha(\dot{s}(t)) dt$, which is finite. Let $\beta = \frac{1}{A}f(x)\alpha$. This form is closed and collinear with α ; by construction, $\text{Supp } \alpha \cap \text{Supp } \beta \neq \emptyset$.

Consider an arbitrary $(n-1)$ -form θ . Since $\text{Supp } \beta \subseteq U$ and $\int_{-\varepsilon}^{\varepsilon} \beta(\dot{s}(t)) dt = 1$,

$$\int_M \theta \wedge \beta = \int_U \theta \wedge \beta = \int_\gamma i^* \theta \int_{-\varepsilon}^{\varepsilon} \beta(\dot{s}(t)) dt = \int_\gamma i^* \theta.$$

where $i : \gamma \hookrightarrow M$ is the inclusion map. This means that $[\beta] = D[\gamma]$ [3, (5.13)].

Given f , the leaf γ constructed in (ii) \Rightarrow (iii) and (iii) \Rightarrow (i) satisfies $f|_\gamma = \text{const}$; then a new β is constructed as above. Given β , the leaf γ constructed as in (iii) \Rightarrow (i) and f constructed in (i) \Rightarrow (ii) satisfy the conditions. \square

Finally, let us show that homological properties of compact leaves are connected with the structure of the cup-product $\smile : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$.

Denote by $c(\alpha)$ the maximum number of homologically independent compact leaves of \mathcal{F}_α . This value plays an important role in the theory of Morse form foliations [14–16]; in particular, in this case

$$c(\alpha) + m(\alpha) \leq \begin{cases} b'_1(M) & [8], \\ b_1(M)/2 & [9], \\ h(M) & [9], \end{cases}$$

where $m(\alpha)$ is the number of minimal components of \mathcal{F}_α , $b'_1(M)$ is the first non-commutative Betti number [1], and $h(M)$ is the maximum rank of a subgroup in $H^1(M, \mathbb{Z})$ with trivial cup-product [17], an important characteristic of the manifold [7].

For a closed one-form α , its *rank* $\text{rk } \alpha$ is the rank of its group of periods:

$$\text{rk } \alpha = \text{rk}_{\mathbb{Q}} \left\{ \int_z \alpha \mid z \in H_1(M) \right\}.$$

Obviously, $0 \leq \text{rk } \alpha \leq b_1(M)$, the first Betti number.

For an arbitrary closed one-form, obviously, $c(\alpha) \leq h(M)$. For forms of the maximum possible rank for the given M , a stronger fact holds (shown in [10] for Morse forms):

T 7.3. Assume $\text{rk } \alpha = b_1(M)$. Then $c(\alpha) \leq \text{rk } \ker \smile$.

P . For $c(\alpha) = 0$ the statement is trivial, so assume there exists a homologically nontrivial compact leaf $\gamma \in \mathcal{F}_\alpha$, $[\gamma] \neq 0$. By Theorem 7.2, there exists a smooth closed one-form $\beta \parallel \alpha$ such that $[\beta] = D[\gamma] \in H^1(M, \mathbb{Z})$. Since $\beta \wedge \alpha = 0$, we have $[\beta] \smile_{\mathbb{R}} [\alpha] = \sum \alpha_i ([\beta] \smile \xi_i) = 0$, where $\smile_{\mathbb{R}}$ is the cup-product on $H^1(M, \mathbb{R})$, $\{\xi_i\}$ is a basis in $H^1(M, \mathbb{Z})$, and $[\alpha] = \sum \alpha_i \xi_i$.

Denote $u_i = [\beta] \smile \xi_i$; $\sum \alpha_i u_i = 0$. Since $\text{rk } \alpha = b_1(M)$, all α_i are independent over \mathbb{Q} . Thus all u_i belong to the torsion of $H^2(M, \mathbb{Z})$, i.e., for some $0 \neq k \in \mathbb{Z}$ we have $k[\beta] \smile \xi_i = 0$ for all i ; so $k[\beta] \in \ker \smile$. Since $H^1(M, \mathbb{Z})$ has no torsion, $k[\beta] \neq 0$.

Now consider $c(\alpha)$ homologically independent compact leaves $\gamma_1, \dots, \gamma_{c(\alpha)}$ and the corresponding β_i as above such that $k_i[\beta_i] \in \ker \smile$ for some $0 \neq K_i \in \mathbb{Z}$. Since the $[\gamma_1], \dots, [\gamma_{c(\alpha)}]$ are independent, so are $[\beta_i] = D[\gamma_i]$, and therefore $k_i[\beta_i]$; thus $\text{rk } \ker \smile \geq c(\alpha)$. \square

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