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On compact leaves of a Morse form foliation

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Abstract. On a compact oriented manifold without boundary, we consider a closed 1-form with singularities of Morse type, called Morse form. We give criteria for the foliation defined by this form to have a compact leaf, to have k homologically independent compact leaves, and to have no minimal components.

1. Introduction and the results

Consider a compact oriented connected smooth n-dimensional manifold M without boundary. On M, consider a smooth differential 1-form ω that is closed, i.e., $d\omega = 0$. By the Poincaré lemma, it is locally the differential of a function: $\omega = df$.

In this paper, we assume f to be a Morse function; then ω is called a *Morse form.* By Morse functions we mean smooth functions with non-degenerate singularities. They are generic (typical) smooth functions: their set is open and dense in the space of smooth functions [7]. Likewise, Morse forms are generic (typical) closed 1-forms: their set is open and dense in the space of all closed 1-forms on M.

Let ω be a Morse form on M. The set of its singularities $\operatorname{Sing}\omega = \{x \in M \mid \omega_x = 0\}$ is finite. On $M \setminus \operatorname{Sing}\omega$ the form ω defines a *foliation* $\mathcal{F}_\omega$ constructed as follows: For any $x \in M \setminus \operatorname{Sing}\omega$, the equation $\{\omega_x(\xi) = 0\}$ defines a distribution of the tangent bundle T_xM. Since ω is closed, this distribution is integrable; its integral surfaces are leaves of $\mathcal{F}_\omega$.

A foliation is a way of slicing the manifold into disjoint submanifolds (called *leaves*) of lower dimension, in our case the dimension $n-1$. This notion is widely

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used in physics. For example, the phase space of a mechanical system is foliated by its energy levels. Foliations of space-time into three-dimensional space-like hypersurfaces have been found to completely characterize the topology of space-time, the singularities describing the topological structure of the gravitational singularities [10].

A foliation $\mathcal{F}_\omega$ has three types of leaves: compact, non-compact compactifiable and non-compact non-compactifiable. If a leaf γ is compactified by $\operatorname{Sing}\omega$, i.e., $\gamma \cup \operatorname{Sing}\omega$ is compact, then it is called *compactifiable*, otherwise it is called *non-compactifiable*. In particular, compact leaves are compactifiable. A foliation is called *compactifiable* if it has only compactifiable leaves, i.e., if it has no *minimal components* (areas covered by non-compactifiable leaves).

Existence of compact leaves and existence of non-compactifiable leaves in a given foliation are classical problems of the foliation theory. We consider both these problems for a Morse form foliation.

Denote by $H_\omega \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$, and by $\mathcal{H} \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $\cdot$ is the cycle intersection and $[\omega] : H_1(M) \to \mathbb{R}$ the integration map. We denote $\operatorname{rk}\omega \equiv \operatorname{rk}_\mathbb{Q} \operatorname{im}[\omega]$.

Melnikova [9] has shown that on a two-dimensional manifold, a foliation $\mathcal{F}_\omega$ is compactifiable iff $\operatorname{rk}\mathcal{H} \geq \operatorname{rk}\omega - 1$. We generalize this fact to arbitrary dimension and give a stronger formulation: $\mathcal{F}_\omega$ is compactifiable iff $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$ (Theorem 8). In Theorem 8 we also show that $\operatorname{rk}\mathcal{H} \leq \operatorname{rk}\omega$, but $\operatorname{rk}\mathcal{H} \neq \operatorname{rk}\omega - 1$.

Farber *et al.* [2], [3] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form. We show that this condition is not a criterion. Then we generalize it to arbitrary (not necessarily transitive) Morse forms and improve it to a criterion.

For this, we introduce the notion of *collinearity* of forms: we call a (not necessarily Morse) smooth closed 1-form α collinear with ω if $\alpha \wedge \omega = 0$; foliations of collinear forms share entire leaves (Proposition 14). We give a criterion for existence of compact leaves: $\mathcal{F}_\omega$ has a compact leaf iff there exists a form $\alpha \not\equiv 0$ collinear with ω such that $[\alpha] \in H^1(M, \mathbb{Z})$ (Theorem 16); what is more, $\mathcal{F}_\omega$ has k homologically independent compact leaves iff there exist k cohomologically independent such forms (Theorem 18).

Finally, we give a condition for compactifiability of $\mathcal{F}_\omega$ in terms of existence of a sufficient number of cohomologically independent forms collinear with ω (Theorem 18).

The paper is organized as follows. In Section 2, we introduce necessary definitions and facts about Morse form foliations. In Section 3, we prove a criterion for a Morse form foliation to be compactifiable. Finally, in Section 4 we introduce a notion of collinearity of 1-forms and use it to give criteria for a foliation to have a compact leaf or k homologically independent compact leaves.

2. Definitions and useful facts

Recall that M is a compact oriented connected smooth n-dimensional manifold without boundary.

2.1. Poincaré duality map. We call an injection $D : H_{n-1}(M) \hookrightarrow H_1(M)$ a Poincaré duality map if there exists a basis $z_i \in H_{n-1}(M)$ such that

$$z_i \cdot Dz_j = \delta_{ij}, \tag{1}$$

where $\cdot$ is the intersection form. For any basis $z_i \in H_{n-1}(M)$, there exists a Poincaré duality map satisfying (1). Obviously, if a subgroup $G \subseteq H_{n-1}(M)$ is a direct summand in $H_{n-1}(M)$, i.e. $H_{n-1}(M) = G \oplus G'$ for some G', then for any basis $z_i \in G$ there exists a Poincaré duality map satisfying (1).

Note that for any subgroup $G \subseteq H_{n-1}(M)$ we have an isomorphism $DG \cong G$; in particular, $\operatorname{rk} DG = \operatorname{rk} G$.

2.2. A Morse form foliation. Recall that for a Morse form ω, the set $\operatorname{Sing}\omega$ is finite since the singularities are isolated and M is compact; on $M \setminus \operatorname{Sing}\omega$ the form defines a foliation $\mathcal{F}_\omega$. The number of its non-compact compactifiable leaves is finite, since each singularity can compactify no more than four leaves. The union of all non-compactifiable leaves is open and has a finite number $m(\omega)$ of connected components [1] called *minimal components*; we call *compactifiable* a foliation that has no minimal components.

For a compact leaf γ there exists an open neighborhood consisting solely of compact leaves: indeed, integrating ω gives near γ a function f with $df = \omega$. Hence, the union of all compact leaves is open. Denote by $H_\omega \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$. A Morse form foliation defines the following decomposition [4]:

$$H_1(M) = DH_\omega \oplus i_*H_1(\Delta), \tag{2}$$

where Δ is the union of all non-compact leaves and singularities, $i : \Delta \hookrightarrow M$, and $D : H_{n-1}(M) \to H_1(M)$ is a Poincaré duality map.

The value $c(\omega) = \operatorname{rk} H_\omega$ is the number of homologically independent compact leaves, i.e. H_ω has a basis of homology classes of compact leaves, $H_\omega = \langle[\gamma_1], \dots, [\gamma_{c(\omega)}]\rangle$ [4]. For a compactifiable foliation, (2) gives

$$c(\omega) \geq \operatorname{rk} \omega, \tag{3}$$

where $\operatorname{rk} \omega = \operatorname{rk} \operatorname{im}[\omega]$, with $[\omega] : H_1(M) \to \mathbb{R}$ being the integration map. Obviously,

$$\operatorname{rk} \omega + \operatorname{rk} \ker[\omega] = b_1(M), \tag{4}$$

the first Betti number.

2.3. Non-commutative Betti number. Arnoux and Levitt [1] denoted by $b_1'(M)$ the *non-commutative Betti number* — the maximal rank (number of free generators) of a free quotient group of $\pi_1(M)$; note that $b_1'(M) \leq b_1(M)$ [8].

Example 1. For an n-dimensional torus we have $b_1'(T^n) = 1$; for the connected sum $\sharp$ of direct products $S^1 \times S^n$, $n > 1$, we have $b_1'\left(\sharp_{i=1}^p (S^1 \times S^n)\right) = p$; for a genus g two-dimensional surface we have $b_1'(M_g^2) = g$ [5].

The topology of the foliation is connected with $b_1'(M)$ [5]:

$$c(\omega) + m(\omega) \leq b_1'(M), \tag{5}$$

where $c(\omega)$ is the number of homologically independent compact leaves and $m(\omega)$ the number of minimal components.

Denote by $h(M)$ the maximum number of cohomologically independent cocycles $u_i \in H^1(M, \mathbb{Z})$ such that the cup-product $u_i \smile u_j = 0$ [4]. Then $c(\omega) \leq h(M)$ [6, Theorem 3.2] and for some Morse form ω on M [5, Theorem 8] it holds

$$c(\omega) = b_1'(M), \tag{6}$$

which gives

$$b_1'(M) \leq h(M). \tag{7}$$

3. Conditions for compactifiability

Denote by $\mathcal{H} \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $H_\omega \subset H_{n-1}(M)$ is the subgroup generated by all compact leaves of $\mathcal{F}_\omega$ and $\cdot$ is the cycle intersection.

Lemma 2. *It holds:*

(i) H_ω *is a direct summand in* $H_{n-1}(M)$*;*

(ii) $\mathcal{H}$ *is a direct summand in* H_ω*;*

(iii) $\mathcal{H}$ *is a direct summand in* $H_{n-1}(M)$*.*

PROOF. It is easy to show that a subgroup of a finitely-generated free abelian group is a direct summand iff its quotient is torsion-free.

(i) Let us show that the quotient $H_{n-1}(M)/H_\omega$ is torsion-free. It has been shown in [4] that there exist compact leaves $\gamma_1, \ldots, \gamma_{c(\omega)} \in \mathcal{F}_\omega$ and closed curves $\alpha_1, \ldots, \alpha_{c(\omega)} \subset M$ such that $[\gamma_i]$ form a basis of H_ω and $[\gamma_i] \cdot [\alpha_j] = \delta_{ij}$. Suppose there exists $0 \neq z = z_0 + H_\omega \in H_{n-1}(M)/H_\omega$ such that $kz = 0$ for some $0 \neq k \in \mathbb{Z}$, i.e., $z_0 \notin H_\omega$ but $kz_0 \in H_\omega$. Then $kz_0 = \sum n_i[\gamma_i]$ and $kz_0 \cdot [\alpha_j] = n_j$. Consider $z_1 = \sum \frac{n_i}{k}[\gamma_i] \in H_\omega$, then $kz_1 = kz_0$. Since $H_\omega \subseteq H_{n-1}(M)$ is torsion-free, we obtain $z_0 = z_1 \in H_\omega$; a contradiction.

(ii) Let us show now that the quotient $H_\omega/\mathcal{H}$ is torsion-free. Similarly, suppose $z \notin \mathcal{H}$, i.e., $z \cdot \ker[\omega] \neq 0$, then $kz \cdot \ker[\omega] \neq 0$ and thus $kz \notin \mathcal{H}$.

(iii) follows from (i) and (ii). $\square$

Recall that $D : H_{n-1}(M) \to H_1(M)$ is a Poincaré duality map defined by the cycle intersection. By Lemma 2, for a basis $z_i \in \mathcal{H}$ there exists a Poincaré duality map that satisfies (1).

Lemma 3. *Let* z_i *be a basis of* $\mathcal{H} \subset H_{n-1}(M)$ *and* D *a corresponding Poincaré duality map. Then the integrals* $\int_{Dz_i} \omega$ *are independent over* $\mathbb{Q}$*.*

Indeed, suppose $\sum n_i \int_{Dz_i} \omega = 0$, i.e., $z = \sum n_i Dz_i \in \ker[\omega]$; then $n_i = z \cdot z_i = 0$.

Proposition 4. *If* $\operatorname{rk}\mathcal{H} \geq \operatorname{rk}\omega - 1$ *then* $\mathcal{F}_\omega$ *is compactifiable.*

In fact we will show below that $\operatorname{rk}\mathcal{H} \neq \operatorname{rk}\omega - 1$, so the above inequality is equivalent to $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$.

PROOF. Consider a basis $z_i \in \mathcal{H}$ and a corresponding Poincaré duality map D. Denote $L_\mathcal{H} = \langle \int_{Dz_i} \omega \rangle$, a linear space over $\mathbb{Q}$; by Lemma 3, $\dim L_\mathcal{H} = \operatorname{rk}\mathcal{H}$.

Suppose that there exists a minimal component U. Then $\operatorname{rk}\omega|_U \geq 2$, i.e., there exist two cycles $s, u \in i_* H_1(U)$, where $i : U \hookrightarrow M$, with independent periods [8]. Denote $L_U = \langle \int_s \omega, \int_u \omega \rangle$, a linear space over $\mathbb{Q}$; $\dim L_U = 2$.

Let us show that $L_\mathcal{H} \cap L_U = 0$. Consider $z = n_s s + n_u u$ such that $\int_z \omega \in L_\mathcal{H}$, i.e. $\int_z \omega = \sum n_i \int_{Dz_i} \omega$. Thus $z - \sum_i n_i Dz_i \in \ker[\omega]$. By definition, $\mathcal{H} \cdot \ker[\omega] = 0$, so $z_j \cdot (z - \sum_i n_i Dz_i) = 0$. Since z_j are generated by compact leaves while $z \in i_* H_1(U)$; we have $z_j \cdot z = 0$. This gives all $n_j = 0$ and thus $\int_z \omega = 0$.

We have $\operatorname{rk}\omega \geq \dim\langle L_\mathcal{H} \cup L_U \rangle = \operatorname{rk}\mathcal{H} + 2$; a contradiction. $\square$

The following condition in terms of compact leaves is geometrically more visual than Proposition 4:

Corollary 5. *Let $\mathcal{F}_\omega$ have $\operatorname{rk}\omega-1$ homologically independent compact leaves γ_i such that $[\gamma_i]\cdot\ker[\omega]=0$. Then $\mathcal{F}_\omega$ is compactifiable, and there exists another compact leaf γ homologically independent from all γ_i.*

Proof. By Proposition 4, the foliation is compactifiable. Then (3) gives $c(\omega)\geq\operatorname{rk}\omega$, so there exists a compact leaf γ such that $[\gamma]\notin\langle[\gamma_i]\rangle$. □

Corollary 5 is not a criterion:

Counterexample 6. On a two-dimensional genus 4 surface M_4^2 represented as a connected sum of four tori T^2, consider a compactifiable foliation such that $\int_{z_1}\omega=\int_{z_2}\omega=1$ and $\int_{z_3}\omega=\int_{z_4}\omega=\sqrt{2}$, so that $\operatorname{rk}\omega=2$ and $c(\omega)=4$; see Figure 1. Then $(z_1-z_2),(z_3-z_4)\in\ker[\omega]$, but for any homologically non-trivial compact leaf $\gamma\in\mathcal{F}_\omega$ we have either $[\gamma]\cdot(z_1-z_2)\neq 0$ or $[\gamma]\cdot(z_3-z_4)\neq 0$, so there are no $\operatorname{rk}\omega-1=1$ homologically independent leaves such that $[\gamma]\cdot\ker\omega=0$. Note that still $\operatorname{rk}\mathcal{H}=2$, cf. Proposition 4.

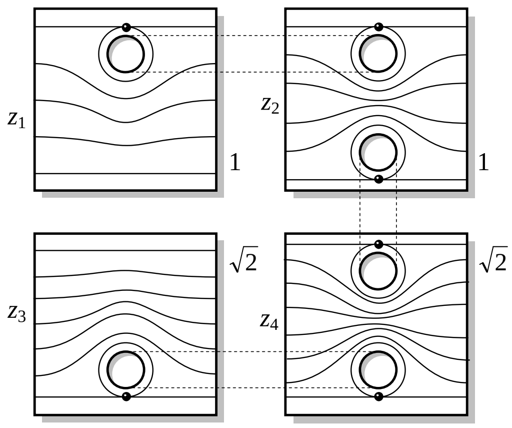

Figure 1. A foliation on a connected sum $M_4^2=\sharp_{i=1}^4 T^2$.

However, with an additional condition the converse to Corollary 5 is true:

Proposition 7. *If $\mathcal{F}_\omega$ is compactifiable and $\operatorname{rk}\omega=c(\omega)$, then there exist $\operatorname{rk}\omega$ homologically independent compact leaves $\gamma_i\in\mathcal{F}_\omega$ such that $[\gamma_i]\cdot\ker[\omega]=0$.*

PROOF. Consider a basis $[\gamma_i] \in H_\omega$. For a compactifiable foliation, Δ mentioned in (2) is the union of a finite number of compactifiable leaves and singularities, so $i_* H_1(\Delta) \subseteq \ker[\omega]$ and $\operatorname{rk}\omega$ is determined by $D[\gamma_i]$. Since $\operatorname{rk}\langle D[\gamma_i]\rangle = \operatorname{rk} H_\omega = c(\omega) = \operatorname{rk}\omega$, all corresponding integrals are rationally independent, so $\ker[\omega] = i_* H_1(\Delta)$. Then $[\gamma_i] \cdot \ker[\omega] = 0$ since $\gamma_i \cap \Delta = \emptyset$. $\square$

In the rest of this section we will study $\operatorname{rk}\mathcal{H}$. By (5), $\operatorname{rk}\mathcal{H} \le b_1'(M)$. The following properties of $\operatorname{rk}\mathcal{H}$ are connected with $\operatorname{rk}\omega$:

Theorem 8. *It holds:*

(i) $\operatorname{rk}\mathcal{H} \le \operatorname{rk}\omega$.

(ii) $\operatorname{rk}\mathcal{H} \ne \operatorname{rk}\omega - 1$.

(iii) $\mathcal{F}_\omega$ *is compactifiable iff* $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$.

PROOF. (i) follows from Lemma 3.

(ii) follows from Proposition 4 and (iii).

(iii) If $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$ then $\mathcal{F}_\omega$ is compactifiable by Proposition 4.

Let now $\mathcal{F}_\omega$ be compactifiable. By Lemma 2 there exists a Poincaré duality map D that satisfies (1) for a basis $[\gamma_i]$ of H_ω. Consider $\varphi : H_\omega \to \mathbb{R}$, $\varphi(z) = \int_{Dz}\omega$. Since Δ in (2) consists of a finite number of compactifiable leaves and singularities, we have $i_* H_1(\Delta) \subseteq \ker[\omega]$; in particular, $\operatorname{rk}\omega = \operatorname{rk}\operatorname{im}\varphi$.

Recall that $\mathcal{H} = \{z \in H_\omega \mid z \cdot \ker[\omega] = 0\}$. Let $u = u_1 + u_2 \in H_1(M)$, $u_1 \in DH_\omega$, $u_2 \in i_* H_1(\Delta)$ according to (2). Since $H_\omega \cdot i_* H_1(\Delta) = 0$, we have $z \cdot u = z \cdot u_1$. For a compactifiable foliation, $u_2 \in \ker[\omega]$, so $u \in \ker[\omega]$ iff $u_1 \in \ker[\omega]$. Thus the above definition can be rewritten as $\mathcal{H} = \{z \in H_\omega \mid z \cdot DH_0 = 0\}$, where $H_0 = \ker\varphi$; in other words, $\mathcal{H}$ is the set of all $z = \sum n_i[\gamma_i]$ such that for all $z_k = \sum m_{kj}[\gamma_j]$ that generate $H_0 \subseteq H_\omega$ it holds $z \cdot Dz_k = 0$, i.e., $\sum n_i m_{ki} = 0$.

The latter linear system implies $\operatorname{rk}\mathcal{H} = \operatorname{rk} H_\omega - \operatorname{rk} H_0$. Since $\operatorname{rk} H_\omega = c(\omega)$ and $\operatorname{rk} H_0 = \operatorname{rk}\ker\varphi = \operatorname{rk} H_\omega - \operatorname{rk}\operatorname{im}\varphi = c(\omega) - \operatorname{rk}\omega$, we obtain $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$. $\square$

Let us consider some special cases.

Corollary 9. *Let* $\ker[\omega] = 0$. *Then* $\mathcal{F}_\omega$ *is compactifiable iff* $c(\omega) = b_1(M)$, *the first Betti number. In this case the cup-product* $\smile : H^1(M,\mathbb{Z}) \times H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$ *is trivial.*

PROOF. Since $\ker[\omega] = 0$, we have $\operatorname{rk}\omega = b_1(M)$ and $\mathcal{H} = H_\omega$. Then the condition $\operatorname{rk}\mathcal{H} = \operatorname{rk}\omega$ from Theorem 8 (iii) is equvalent to $c(\omega) = b_1(M)$ and thus $H_\omega = H_{n-1}(M)$. Then $H^1(M,\mathbb{Z}) = \langle z_i\rangle$, where z_i are cocycles dual to $[\gamma_i]$, a basis of H_ω, and $\gamma_i \cap \gamma_j = \emptyset$ implies $z_i \smile z_j = 0$. $\square$

So if $\ker[\omega] = 0$ and $\smile \not\equiv 0$, then $\mathcal{F}_\omega$ has a minimal component. If, however, $\smile \equiv 0$, then both cases are possible. Indeed, on the one hand, in any cohomology class $[\omega]$, $\operatorname{rk}\omega > 1$, there exists a Morse form with minimal foliation [1]. On the other hand, the foliation can be compactifiable:

Example 10. Consider a connected sum $M = \sharp_{i=1}^p (S^1 \times S^n)_i$, $n > 1$; see Figure 2. Then $b_1(M) = b_1'(M) = p$ (Example 1), which by (7) gives $\smile \equiv 0$. Consider ω given on each $(S^1 \times S^n)_i$ by $\omega_i = \alpha_i dt$, where t is a coordinate on S^1 and all $\alpha_i \in \mathbb{R}$ are independent over $\mathbb{Q}$ so that $\operatorname{rk}\omega = p$. Obviously, $\mathcal{F}_\omega$ is compactifiable (its compact leaves are spheres S^n).

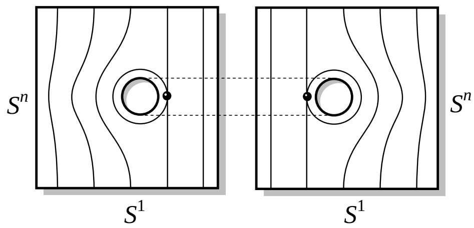

Figure 2. A foliation on a connected sum $(S^1 \times S^n) \sharp (S^1 \times S^n)$.

Corollary 11. *For a two-dimensional genus g surface M_g^2 it holds*

$$\operatorname{rk}\mathcal{H} \leq \operatorname{rk}\omega \leq 2g - c(\omega) \leq 2g - \operatorname{rk}\mathcal{H}. \tag{8}$$

If $\operatorname{rk}\mathcal{H} = g$, then $\mathcal{F}_\omega$ is compactifiable.

Proof. The lower bound is by Theorem 8 (i). Since leaves are one-dimensional, $\mathcal{H} \subseteq H_\omega \subseteq \ker[\omega]$ and $\operatorname{rk}\ker[\omega] = 2g - \operatorname{rk}\omega$ gives the upper bound. If $\operatorname{rk}\mathcal{H} = g$ then (8) implies $\operatorname{rk}\omega = g$ and $\mathcal{F}_\omega$ is compactifiable by Theorem 8 (iii). $\square$

4. Criterion for the presence of compact leaves

Farber *et al.* proved a necessary condition of existence of a compact leaf γ in terms of zero cup-product:

Proposition 12 ([2, Proposition 9.14],[3, Proposition 3])**.** *For so-called transitive Morse forms, if $\mathcal{F}_\omega$ has a compact leaf with $[\gamma] \neq 0$ then there exists a smooth closed 1-form α, $0 \neq [\alpha] \in H^1(M, \mathbb{Z})$, such that $[\alpha] \smile [\omega] = 0$.*

The converse is, however, not true; see Counterexample 17 below. Moreover, no sufficient conditions for existing of a compact leaf can be given in cohomologous terms: any cohomology class $[\omega]$, $\operatorname{rk}\omega > 1$, contains a form with minimal foliation [1].

We call 1-forms α and β *collinear* if $\alpha\wedge\beta = 0$. Using the notion of collinearity instead of zero cup-product, we will generalize Proposition 12 to an arbitrary (not necessarily transitive) Morse form and refine it to a criterion. For closed 1-forms the equation $\alpha \wedge \beta = 0$ implies $[\alpha] \smile [\beta] = 0$ but not *vice versa*, so collinearity is a stronger condition.

Denote $\operatorname{Supp}\alpha = M \setminus \operatorname{Sing}\alpha$. If α is closed, on $\operatorname{Supp}\alpha$ the integrable distribution $\{\alpha = 0\}$ defines a foliation $\mathcal{F}_\alpha$.

Lemma 13. *For closed collinear 1-forms α, β, on $\operatorname{Supp}\beta$ it holds $\alpha = f(x)\beta$, where $f(x)$ is constant on leaves of $\mathcal{F}_\beta$. In particular, on $\operatorname{Supp}\alpha \cap \operatorname{Supp}\beta$ it holds $\mathcal{F}_\alpha = \mathcal{F}_\beta$.*

Proof. On $\operatorname{Supp}\beta$ there exists a smooth vector field ξ_x such that $\beta(\xi_x) \neq 0$. Consider $f(x) = \frac{\alpha(\xi_x)}{\beta(\xi_x)}$, which is well-defined: for any vector fields ξ_x, η_x we have $\alpha(\xi_x)\beta(\eta_x) - \alpha(\eta_x)\beta(\xi_x) = (\alpha \wedge \beta)(\xi_x, \eta_x) = 0$. Thus on $\operatorname{Supp}\beta$ we have $\alpha = f(x)\beta$.

Since α and β are closed, $df \wedge \beta = d\alpha - f d\beta = 0$. Consider a vector field ξ tangent to the leaves of $\mathcal{F}_\beta$ and η normal to the leaves. Then $df \wedge \beta(\xi, \eta) = 0$ implies $(df)(\xi) = 0$, i.e. f is constant on leaves. □

Proposition 14. *Let α be a smooth closed 1-form collinear with a Morse form ω; $\alpha \not\equiv 0$ and $[\alpha] \in H^1(M, \mathbb{Z})$. Then $\operatorname{Supp}\alpha$ is the union of a non-empty subset of compact leaves of $\mathcal{F}_\omega$ and a subset of compactifiable leaves of $\mathcal{F}_\omega$. These leaves of $\mathcal{F}_\omega$ are leaves of $\mathcal{F}_\alpha$.*

Proof. All leaves of $\mathcal{F}_\alpha$ are closed. Indeed, since $[\alpha] \in H^1(M, \mathbb{Z})$, it defines a smooth map $F_{[\alpha]} : M \to S^1$,

$$F_{[\alpha]}(x) = e^{2\pi i \int_{x_0}^{x} \alpha}.$$

Obviously, $F_{[\alpha]}$ is constant on leaves of $\mathcal{F}_\alpha$ and the critical set of $F_{[\alpha]}$ coincides with $\operatorname{Sing}\alpha$. So on $\operatorname{Supp}\alpha$ the map is regular and by the implicit function theorem each leaf of $\mathcal{F}_\alpha$ (which is a connected component of a level $F_{[\alpha]}^{-1}(y)$, $y \in S^1$) is a closed codimension-one submanifold of $\operatorname{Supp}\alpha$ (not necessarily closed in M).

Next, if for a leaf $\gamma \in \mathcal{F}_\omega$ it holds $\gamma \cap \operatorname{Supp}\alpha \neq \emptyset$ then $\gamma \subseteq \operatorname{Supp}\alpha$. Indeed, suppose there exists $x_0 \in \gamma \cap \operatorname{Sing}\alpha$. By Lemma 13, on $\operatorname{Supp}\omega$ it holds $\alpha = f(x)\omega$, where the function $f(x)$ is constant on leaves. Since $x_0 \in \operatorname{Supp}\omega$, we have $f(x_0) = 0$ and so $f|_\gamma = 0$, which gives $\gamma \cap \operatorname{Supp}\alpha = \emptyset$; a contradiction.

Similarly, if for a leaf $\gamma \in \mathcal{F}_\alpha$ it holds $\gamma \cap \operatorname{Sing}\omega \neq \emptyset$ then $\gamma \subseteq \operatorname{Sing}\omega$. However, since $\operatorname{Sing}\omega$ consists of isolated points, such a leaf γ would be a point. This gives $\operatorname{Supp}\alpha \cap \operatorname{Sing}\omega = \emptyset$ and thus $\operatorname{Supp}\alpha \subseteq \operatorname{Supp}\omega$.

Now Lemma 13 implies that all leaves of $\mathcal{F}_\alpha$ are leaves of $\mathcal{F}_\omega$. Since all leaves of $\mathcal{F}_\alpha$ are closed in $\operatorname{Supp}\alpha$, the latter cannot contain any non-compactifiable leaves of $\mathcal{F}_\omega$. It cannot consist solely of non-compact compactifiable leaves of $\mathcal{F}_\omega$ since their number is finite while $\operatorname{Supp}\alpha$ is open. Thus it must contain compact leaves of $\mathcal{F}_\omega$. $\square$

Lemma 15. *In the conditions of Proposition 14, if* $[\alpha] \neq 0$ *then* $\mathcal{F}_\alpha$ *has a compact leaf with* $[\gamma] \neq 0$.

Proof. Following the reasoning of [4] it is easy to show that (2) holds for α even though it is not a Morse form. Since its Δ consists of $\operatorname{Sing}\alpha$ and a finite number of compactifiable leaves, $\operatorname{rk}\alpha$ is determined by DH_α. However, if $[\gamma] = 0$ for any compact $\gamma \in \mathcal{F}_\alpha$ then $H_\alpha = 0$ and thus $\operatorname{rk}\alpha = 0$, i.e., $[\alpha] = 0$. $\square$

Now we are ready to proof the main result of this section: a criterion for existence of a compact leaf.

Theorem 16. *The following conditions are equivalent:*

(i) $\mathcal{F}_\omega$ *has a compact leaf* γ*;*

(ii) *There exists a smooth function* $f(x) \not\equiv \mathrm{const}$ *such that* df *is collinear with* ω*;*

(iii) *There exists a smooth closed 1-form* $\alpha \not\equiv 0$, $[\alpha] \in H^1(M, \mathbb{Z})$*, collinear with* ω.

Moreover, γ *can be chosen with* $[\gamma] \neq 0$ *iff* α *can be chosen with* $[\alpha] \neq 0$.

Note that f and α are not required to be of Morse type.

Proof. (i) $\Rightarrow$ (ii), (iii): Let γ be a compact leaf. Consider a cylindrical neighborhood $\mathcal{O}(\gamma) = \gamma \times I$ consisting of diffeomorphic leaves. Let $(x^1, \ldots, x^n)$ be local coordinates in $\mathcal{O}(\gamma)$ such that $(x^1, \ldots, x^{n-1})$ are coordinates in γ and x^n in I. Consider a smooth function $f(x) = f(x^n) \not\equiv \mathrm{const}$ in $\mathcal{O}(\gamma)$ and $f(x) = 0$ on $M \setminus \mathcal{O}(\gamma)$. Let $x \in \mathcal{O}(\gamma)$; consider the leaf $\gamma' \ni x$. Let $\eta_1, \eta_2 \in T_xM$; then $\eta_i = \xi_i + a_i n$, where $\xi_i \in T_x\gamma'$, $a_i \in \mathbb{R}$, and $n \in T_xM \setminus T_x\gamma'$. We obtain $df(\eta_i) = a_i df(n)$ and $\omega(\eta_i) = a_i\omega(n)$. Thus $df \wedge \omega(\eta_1, \eta_2) = 0$, which proves (ii).

Consider now $\alpha = f(x)\omega$; obviously, α is closed and collinear with ω. In addition, we can choose f such that $[\alpha] \in H^1(M, \mathbb{Z})$, which proves (iii). Finally, if $[\gamma] \neq 0$ then there exists a cycle $z \in H_1(M)$ such that $z \cdot [\gamma] = 1$; choosing f non-negative we obtain $\int_z \alpha \neq 0$, thus $[\alpha] \neq 0$.

(ii), (iii) $\Rightarrow$ (i): This has been shown as Proposition 14 and Lemma 15. $\square$

Now Proposition 12 follows from Theorem 16. What is more, the same theorem shows that Proposition 12 is not a criterion:

Counterexample 17. The converse to Proposition 12 is not true for manifolds with $b_1'(M) > 1$; see Section 2.3. Indeed, by (6) there exists a Morse form ω on M such that $c(\omega) = b_1'(M)$. By Theorem 16 there exists a form α, $0 \neq [\alpha] \in H^1(M,\mathbb{Z})$, such that $\alpha \wedge \omega = 0$ and thus $[\alpha] \smile [\omega] = 0$. The same foliation $\mathcal{F}_\omega$ can be defined by a form of rank $b_1'(M)$ [6, Theorem 4.1], so we can assume that $\operatorname{rk}\omega = b_1'(M) > 1$. Then there exists a form ω' with a minimal foliation and $[\omega'] = [\omega]$ [1]; in particular, $[\alpha] \smile [\omega'] = 0$.

Recall that $c(\omega) = \operatorname{rk} H_\omega$ is the total number of homologically independent compact leaves of $\mathcal{F}_\omega$. Theorem 16 states that $c(\omega) \neq 0$ iff there is a suitable $[\alpha] \neq 0$. This can be easily generalized to an arbitrary number k: $c(\omega) \geq k$ iff there are k independent α's, which gives a criterion for existence of k homologically independent compact leaves:

Theorem 18. *The following conditions are equivalent:*

(i) *$\mathcal{F}_\omega$ has k homologically independent compact leaves γ_i;*

(ii) *There exist k cohomologically independent smooth closed 1-forms α_i, $[\alpha_i] \in H^1(M,\mathbb{Z})$, collinear with ω.*

If the above conditions hold for $k = b_1'(M)$ then $\mathcal{F}_\omega$ is compactifiable.

Proof. (i) $\Rightarrow$ (ii): For each γ_i construct a form α_i, $[\alpha_i] \neq 0$, as in Theorem 16. Consider a Poincaré duality map D that satisfies (1) for γ_i. Since $\int_{D\gamma_i} \alpha_j = \delta_{ij}$, all $[\alpha_i]$ are independent.

(ii) $\Rightarrow$ (i): As has been noted in Lemma 15, $\operatorname{rk}\alpha_i$ is determined by DH_{α_i}. By Proposition 14 we have $H_{\alpha_i} \subseteq H_\omega$ and thus the rank of the whole system $\langle [\alpha_1], \ldots, [\alpha_k] \rangle$ is determined by H_ω, which implies $c(\omega) = \operatorname{rk} H_\omega \geq k$.

Finally, by (5), $c(\omega) \geq k = b_1'(M)$ implies $m(\omega) = 0$, i.e. $\mathcal{F}_\omega$ is compactifiable. $\square$

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