

## On compact leaves of a Morse form foliation

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**Abstract.** On a compact oriented manifold without boundary, we consider a closed 1-form with singularities of Morse type, called Morse form. We give criteria for the foliation defined by this form to have a compact leaf, to have  $k$  homologically independent compact leaves, and to have no minimal components.

### 1. Introduction and the results

Consider a compact oriented connected smooth  $n$ -dimensional manifold  $M$  without boundary. On  $M$ , consider a smooth differential 1-form  $\omega$  that is closed, i.e.,  $d\omega = 0$ . By the Poincaré lemma, it is locally the differential of a function:  $\omega = df$ .

In this paper, we assume  $f$  to be a Morse function; then  $\omega$  is called a *Morse form*. By Morse functions we mean smooth functions with non-degenerate singularities. They are generic (typical) smooth functions: their set is open and dense in the space of smooth functions [7]. Likewise, Morse forms are generic (typical) closed 1-forms: their set is open and dense in the space of all closed 1-forms on  $M$ .

Let  $\omega$  be a Morse form on  $M$ . The set of its singularities  $\text{Sing } \omega = \{x \in M \mid \omega_x = 0\}$  is finite. On  $M \setminus \text{Sing } \omega$  the form  $\omega$  defines a *foliation*  $\mathcal{F}_\omega$  constructed as follows: For any  $x \in M \setminus \text{Sing } \omega$ , the equation  $\{\omega_x(\xi) = 0\}$  defines a distribution of the tangent bundle  $T_x M$ . Since  $\omega$  is closed, this distribution is integrable; its integral surfaces are leaves of  $\mathcal{F}_\omega$ .

A foliation is a way of slicing the manifold into disjoint submanifolds (called *leaves*) of lower dimension, in our case the dimension  $n - 1$ . This notion is widely

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used in physics. For example, the phase space of a mechanical system is foliated by its energy levels. Foliations of space-time into three-dimensional space-like hypersurfaces have been found to completely characterize the topology of space-time, the singularities describing the topological structure of the gravitational singularities [10].

A foliation  $\mathcal{F}_\omega$  has three types of leaves: compact, non-compact compactifiable and non-compact non-compactifiable. If a leaf  $\gamma$  is compactified by  $\text{Sing } \omega$ , i.e.,  $\gamma \cup \text{Sing } \omega$  is compact, then it is called *compactifiable*, otherwise it is called *non-compactifiable*. In particular, compact leaves are compactifiable. A foliation is called *compactifiable* if it has only compactifiable leaves, i.e., if it has no *minimal components* (areas covered by non-compactifiable leaves).

Existence of compact leaves and existence of non-compactifiable leaves in a given foliation are classical problems of the foliation theory. We consider both these problems for a Morse form foliation.

Denote by  $H_\omega \subseteq H_{n-1}(M)$  a group generated by all compact leaves of  $\mathcal{F}_\omega$ , and by  $\mathcal{H} \subseteq H_\omega$  a subgroup of all  $z \in H_\omega$  such that  $z \cdot \ker[\omega] = 0$ , where  $\cdot$  is the cycle intersection and  $[\omega] : H_1(M) \rightarrow \mathbb{R}$  the integration map. We denote  $\text{rk } \omega \equiv \text{rk}_{\mathbb{Q}} \text{im}[\omega]$ .

MELNIKOVA [9] has shown that on a two-dimensional manifold, a foliation  $\mathcal{F}_\omega$  is compactifiable iff  $\text{rk } \mathcal{H} \geq \text{rk } \omega - 1$ . We generalize this fact to arbitrary dimension and give a stronger formulation:  $\mathcal{F}_\omega$  is compactifiable iff  $\text{rk } \mathcal{H} = \text{rk } \omega$  (Theorem 8). In Theorem 8 we also show that  $\text{rk } \mathcal{H} \leq \text{rk } \omega$ , but  $\text{rk } \mathcal{H} \neq \text{rk } \omega - 1$ .

FARBER *et al.* [2], [3] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form. We show that this condition is not a criterion. Then we generalize it to arbitrary (not necessarily transitive) Morse forms and improve it to a criterion.

For this, we introduce the notion of *collinearity* of forms: we call a (not necessarily Morse) smooth closed 1-form  $\alpha$  collinear with  $\omega$  if  $\alpha \wedge \omega = 0$ ; foliations of collinear forms share entire leaves (Proposition 14). We give a criterion for existence of compact leaves:  $\mathcal{F}_\omega$  has a compact leaf iff there exists a form  $\alpha \neq 0$  collinear with  $\omega$  such that  $[\alpha] \in H^1(M, \mathbb{Z})$  (Theorem 16); what is more,  $\mathcal{F}_\omega$  has  $k$  homologically independent compact leaves iff there exist  $k$  cohomologically independent such forms (Theorem 18).

Finally, we give a condition for compactifiability of  $\mathcal{F}_\omega$  in terms of existence of a sufficient number of cohomologically independent forms collinear with  $\omega$  (Theorem 18).

The paper is organized as follows. In Section 2, we introduce necessary definitions and facts about Morse form foliations. In Section 3, we prove a criterion

for a Morse form foliation to be compactifiable. Finally, in Section 4 we introduce a notion of collinearity of 1-forms and use it to give criteria for a foliation to have a compact leaf or  $k$  homologically independent compact leaves.

## 2. Definitions and useful facts

Recall that  $M$  is a compact oriented connected smooth  $n$ -dimensional manifold without boundary.

**2.1. Poincaré duality map.** We call an injection  $D : H_{n-1}(M) \hookrightarrow H_1(M)$  a Poincaré duality map if there exists a basis  $z_i \in H_{n-1}(M)$  such that

$$z_i \cdot Dz_j = \delta_{ij}, \quad (1)$$

where  $\cdot$  is the intersection form. For any basis  $z_i \in H_{n-1}(M)$ , there exists a Poincaré duality map satisfying (1). Obviously, if a subgroup  $G \subseteq H_{n-1}(M)$  is a direct summand in  $H_{n-1}(M)$ , i.e.  $H_{n-1}(M) = G \oplus G'$  for some  $G'$ , then for any basis  $z_i \in G$  there exists a Poincaré duality map satisfying (1).

Note that for any subgroup  $G \subseteq H_{n-1}(M)$  we have an isomorphism  $DG \cong G$ ; in particular,  $\text{rk } DG = \text{rk } G$ .

**2.2. A Morse form foliation.** Recall that for a Morse form  $\omega$ , the set  $\text{Sing } \omega$  is finite since the singularities are isolated and  $M$  is compact; on  $M \setminus \text{Sing } \omega$  the form defines a foliation  $\mathcal{F}_\omega$ . The number of its non-compact compactifiable leaves is finite, since each singularity can compactify no more than four leaves. The union of all non-compactifiable leaves is open and has a finite number  $m(\omega)$  of connected components [1] called *minimal components*; we call *compactifiable* a foliation that has no minimal components.

For a compact leaf  $\gamma$  there exists an open neighborhood consisting solely of compact leaves: indeed, integrating  $\omega$  gives near  $\gamma$  a function  $f$  with  $df = \omega$ . Hence, the union of all compact leaves is open. Denote by  $H_\omega \subseteq H_{n-1}(M)$  a group generated by all compact leaves of  $\mathcal{F}_\omega$ . A Morse form foliation defines the following decomposition [4]:

$$H_1(M) = DH_\omega \oplus i_*H_1(\Delta), \quad (2)$$

where  $\Delta$  is the union of all non-compact leaves and singularities,  $i : \Delta \hookrightarrow M$ , and  $D : H_{n-1}(M) \rightarrow H_1(M)$  is a Poincaré duality map.

The value  $c(\omega) = \text{rk } H_\omega$  is the number of homologically independent compact leaves, i.e.  $H_\omega$  has a basis of homology classes of compact leaves,  $H_\omega = \langle [\gamma_1], \dots, [\gamma_{c(\omega)}] \rangle$  [4]. For a compactifiable foliation, (2) gives

$$c(\omega) \geq \text{rk } \omega, \quad (3)$$

where  $\text{rk } \omega = \text{rk } \text{im}[\omega]$ , with  $[\omega] : H_1(M) \rightarrow \mathbb{R}$  being the integration map. Obviously,

$$\text{rk } \omega + \text{rk } \ker[\omega] = b_1(M), \quad (4)$$

the first Betti number.

**2.3. Non-commutative Betti number.** ARNOUX and LEVITT [1] denoted by  $b'_1(M)$  the *non-commutative Betti number* — the maximal rank (number of free generators) of a free quotient group of  $\pi_1(M)$ ; note that  $b'_1(M) \leq b_1(M)$  [8].

*Example 1.* For an  $n$ -dimensional torus we have  $b'_1(T^n) = 1$ ; for the connected sum  $\sharp$  of direct products  $S^1 \times S^n$ ,  $n > 1$ , we have  $b'_1(\sharp_{i=1}^p (S^1 \times S^n)) = p$ ; for a genus  $g$  two-dimensional surface we have  $b'_1(M_g^2) = g$  [5].

The topology of the foliation is connected with  $b'_1(M)$  [5]:

$$c(\omega) + m(\omega) \leq b'_1(M), \quad (5)$$

where  $c(\omega)$  is the number of homologically independent compact leaves and  $m(\omega)$  the number of minimal components.

Denote by  $h(M)$  the maximum number of cohomologically independent cocycles  $u_i \in H^1(M, \mathbb{Z})$  such that the cup-product  $u_i \smile u_j = 0$  [4]. Then  $c(\omega) \leq h(M)$  [6, Theorem 3.2] and for some Morse form  $\omega$  on  $M$  [5, Theorem 8] it holds

$$c(\omega) = b'_1(M), \quad (6)$$

which gives

$$b'_1(M) \leq h(M). \quad (7)$$

### 3. Conditions for compactifiability

Denote by  $\mathcal{H} \subseteq H_\omega$  a subgroup of all  $z \in H_\omega$  such that  $z \cdot \ker[\omega] = 0$ , where  $H_\omega \subset H_{n-1}(M)$  is the subgroup generated by all compact leaves of  $\mathcal{F}_\omega$  and  $\cdot$  is the cycle intersection.

**Lemma 2.** *It holds:*

- (i)  $H_\omega$  is a direct summand in  $H_{n-1}(M)$ ;
- (ii)  $\mathcal{H}$  is a direct summand in  $H_\omega$ ;
- (iii)  $\mathcal{H}$  is a direct summand in  $H_{n-1}(M)$ .

PROOF. It is easy to show that a subgroup of a finitely-generated free abelian group is a direct summand iff its quotient is torsion-free.

(i) Let us show that the quotient  $H_{n-1}(M)/H_\omega$  is torsion-free. It has been shown in [4] that there exist compact leaves  $\gamma_1, \dots, \gamma_{c(\omega)} \in \mathcal{F}_\omega$  and closed curves  $\alpha_1, \dots, \alpha_{c(\omega)} \subset M$  such that  $[\gamma_i]$  form a basis of  $H_\omega$  and  $[\gamma_i] \cdot [\alpha_j] = \delta_{ij}$ . Suppose there exists  $0 \neq z = z_0 + H_\omega \in H_{n-1}(M)/H_\omega$  such that  $kz = 0$  for some  $0 \neq k \in \mathbb{Z}$ , i.e.,  $z_0 \notin H_\omega$  but  $kz_0 \in H_\omega$ . Then  $kz_0 = \sum n_i [\gamma_i]$  and  $kz_0 \cdot [\alpha_j] = n_j$ . Consider  $z_1 = \sum \frac{n_i}{k} [\gamma_i] \in H_\omega$ , then  $kz_1 = kz_0$ . Since  $H_\omega \subseteq H_{n-1}(M)$  is torsion-free, we obtain  $z_0 = z_1 \in H_\omega$ ; a contradiction.

(ii) Let us show now that the quotient  $H_\omega/\mathcal{H}$  is torsion-free. Similarly, suppose  $z \notin \mathcal{H}$ , i.e.,  $z \cdot \ker[\omega] \neq 0$ , then  $kz \cdot \ker[\omega] \neq 0$  and thus  $kz \notin \mathcal{H}$ .

(iii) follows from (i) and (ii).  $\square$

Recall that  $D : H_{n-1}(M) \rightarrow H_1(M)$  is a Poincaré duality map defined by the cycle intersection. By Lemma 2, for a basis  $z_i \in \mathcal{H}$  there exists a Poincaré duality map that satisfies (1).

**Lemma 3.** *Let  $z_i$  be a basis of  $\mathcal{H} \subset H_{n-1}(M)$  and  $D$  a corresponding Poincaré duality map. Then the integrals  $\int_{Dz_i} \omega$  are independent over  $\mathbb{Q}$ .*

Indeed, suppose  $\sum n_i \int_{Dz_i} \omega = 0$ , i.e.,  $z = \sum n_i Dz_i \in \ker[\omega]$ ; then  $n_i = z \cdot z_i = 0$ .

**Proposition 4.** *If  $\text{rk } \mathcal{H} \geq \text{rk } \omega - 1$  then  $\mathcal{F}_\omega$  is compactifiable.*

In fact we will show below that  $\text{rk } \mathcal{H} \neq \text{rk } \omega - 1$ , so the above inequality is equivalent to  $\text{rk } \mathcal{H} = \text{rk } \omega$ .

PROOF. Consider a basis  $z_i \in \mathcal{H}$  and a corresponding Poincaré duality map  $D$ . Denote  $L_{\mathcal{H}} = \langle \int_{Dz_i} \omega \rangle$ , a linear space over  $\mathbb{Q}$ ; by Lemma 3,  $\dim L_{\mathcal{H}} = \text{rk } \mathcal{H}$ .

Suppose that there exists a minimal component  $U$ . Then  $\text{rk } \omega|_U \geq 2$ , i.e., there exist two cycles  $s, u \in i_* H_1(U)$ , where  $i : U \hookrightarrow M$ , with independent periods [8]. Denote  $L_U = \langle \int_s \omega, \int_u \omega \rangle$ , a linear space over  $\mathbb{Q}$ ;  $\dim L_U = 2$ .

Let us show that  $L_{\mathcal{H}} \cap L_U = 0$ . Consider  $z = n_s s + n_u u$  such that  $\int_z \omega \in L_{\mathcal{H}}$ , i.e.  $\int_z \omega = \sum n_i \int_{Dz_i} \omega$ . Thus  $z - \sum_i n_i Dz_i \in \ker[\omega]$ . By definition,  $\mathcal{H} \cdot \ker[\omega] = 0$ , so  $z_j \cdot (z - \sum_i n_i Dz_i) = 0$ . Since  $z_j$  are generated by compact leaves while  $z \in i_* H_1(U)$ ; we have  $z_j \cdot z = 0$ . This gives all  $n_j = 0$  and thus  $\int_z \omega = 0$ .

We have  $\text{rk } \omega \geq \dim \langle L_{\mathcal{H}} \cup L_U \rangle = \text{rk } \mathcal{H} + 2$ ; a contradiction.  $\square$

The following condition in terms of compact leaves is geometrically more visual than Proposition 4:

**Corollary 5.** *Let  $\mathcal{F}_\omega$  have  $\text{rk } \omega - 1$  homologically independent compact leaves  $\gamma_i$  such that  $[\gamma_i] \cdot \ker[\omega] = 0$ . Then  $\mathcal{F}_\omega$  is compactifiable, and there exists another compact leaf  $\gamma$  homologically independent from all  $\gamma_i$ .*

PROOF. By Proposition 4, the foliation is compactifiable. Then (3) gives  $c(\omega) \geq \text{rk } \omega$ , so there exists a compact leaf  $\gamma$  such that  $[\gamma] \notin \langle [\gamma_i] \rangle$ .  $\square$

Corollary 5 is not a criterion:

*Counterexample 6.* On a two-dimensional genus 4 surface  $M_4^2$  represented as a connected sum of four tori  $T^2$ , consider a compactifiable foliation such that  $\int_{z_1} \omega = \int_{z_2} \omega = 1$  and  $\int_{z_3} \omega = \int_{z_4} \omega = \sqrt{2}$ , so that  $\text{rk } \omega = 2$  and  $c(\omega) = 4$ ; see Figure 1. Then  $(z_1 - z_2), (z_3 - z_4) \in \ker[\omega]$ , but for any homologically non-trivial compact leaf  $\gamma \in \mathcal{F}_\omega$  we have either  $[\gamma] \cdot (z_1 - z_2) \neq 0$  or  $[\gamma] \cdot (z_3 - z_4) \neq 0$ , so there are no  $\text{rk } \omega - 1 = 1$  homologically independent leaves such that  $[\gamma] \cdot \ker \omega = 0$ . Note that still  $\text{rk } \mathcal{H} = 2$ , cf. Proposition 4.

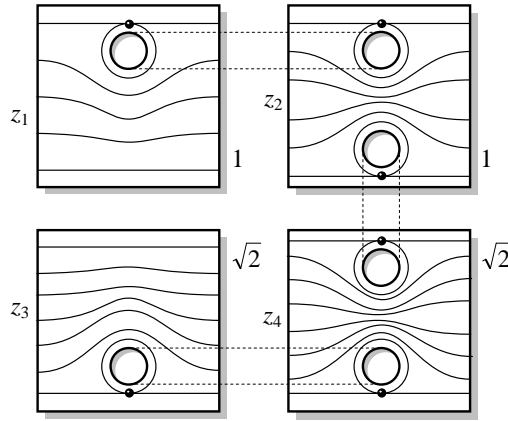


Figure 1. A foliation on a connected sum  $M_4^2 = \#_{i=1}^4 T^2$ .

However, with an additional condition the converse to Corollary 5 is true:

**Proposition 7.** *If  $\mathcal{F}_\omega$  is compactifiable and  $\text{rk } \omega = c(\omega)$ , then there exist  $\text{rk } \omega$  homologically independent compact leaves  $\gamma_i \in \mathcal{F}_\omega$  such that  $[\gamma_i] \cdot \ker[\omega] = 0$ .*

PROOF. Consider a basis  $[\gamma_i] \in H_\omega$ . For a compactifiable foliation,  $\Delta$  mentioned in (2) is the union of a finite number of compactifiable leaves and singularities, so  $i_*H_1(\Delta) \subseteq \ker[\omega]$  and  $\text{rk } \omega$  is determined by  $D[\gamma_i]$ . Since  $\text{rk} \langle D[\gamma_i] \rangle = \text{rk } H_\omega = c(\omega) = \text{rk } \omega$ , all corresponding integrals are rationally independent, so  $\ker[\omega] = i_*H_1(\Delta)$ . Then  $[\gamma_i] \cdot \ker[\omega] = 0$  since  $\gamma_i \cap \Delta = \emptyset$ .  $\square$

In the rest of this section we will study  $\text{rk } \mathcal{H}$ . By (5),  $\text{rk } \mathcal{H} \leq b'_1(M)$ . The following properties of  $\text{rk } \mathcal{H}$  are connected with  $\text{rk } \omega$ :

**Theorem 8.** *It holds:*

- (i)  $\text{rk } \mathcal{H} \leq \text{rk } \omega$ .
- (ii)  $\text{rk } \mathcal{H} \neq \text{rk } \omega - 1$ .
- (iii)  $\mathcal{F}_\omega$  is compactifiable iff  $\text{rk } \mathcal{H} = \text{rk } \omega$ .

PROOF. (i) follows from Lemma 3.

(ii) follows from Proposition 4 and (iii).

(iii) If  $\text{rk } \mathcal{H} = \text{rk } \omega$  then  $\mathcal{F}_\omega$  is compactifiable by Proposition 4.

Let now  $\mathcal{F}_\omega$  be compactifiable. By Lemma 2 there exists a Poincaré duality map  $D$  that satisfies (1) for a basis  $[\gamma_i]$  of  $H_\omega$ . Consider  $\varphi : H_\omega \rightarrow \mathbb{R}$ ,  $\varphi(z) = \int_{Dz} \omega$ . Since  $\Delta$  in (2) consists of a finite number of compactifiable leaves and singularities, we have  $i_*H_1(\Delta) \subseteq \ker[\omega]$ ; in particular,  $\text{rk } \omega = \text{rk } \text{im } \varphi$ .

Recall that  $\mathcal{H} = \{z \in H_\omega \mid z \cdot \ker[\omega] = 0\}$ . Let  $u = u_1 + u_2 \in H_1(M)$ ,  $u_1 \in DH_\omega$ ,  $u_2 \in i_*H_1(\Delta)$  according to (2). Since  $H_\omega \cdot i_*H_1(\Delta) = 0$ , we have  $z \cdot u = z \cdot u_1$ . For a compactifiable foliation,  $u_2 \in \ker[\omega]$ , so  $u \in \ker[\omega]$  iff  $u_1 \in \ker[\omega]$ . Thus the above definition can be rewritten as  $\mathcal{H} = \{z \in H_\omega \mid z \cdot DH_0 = 0\}$ , where  $H_0 = \ker \varphi$ ; in other words,  $\mathcal{H}$  is the set of all  $z = \sum n_i [\gamma_i]$  such that for all  $z_k = \sum m_{kj} [\gamma_j]$  that generate  $H_0 \subseteq H_\omega$  it holds  $z \cdot Dz_k = 0$ , i.e.,  $\sum n_i m_{ki} = 0$ .

The latter linear system implies  $\text{rk } \mathcal{H} = \text{rk } H_\omega - \text{rk } H_0$ . Since  $\text{rk } H_\omega = c(\omega)$  and  $\text{rk } H_0 = \text{rk } \ker \varphi = \text{rk } H_\omega - \text{rk } \text{im } \varphi = c(\omega) - \text{rk } \omega$ , we obtain  $\text{rk } \mathcal{H} = \text{rk } \omega$ .  $\square$

Let us consider some special cases.

**Corollary 9.** *Let  $\ker[\omega] = 0$ . Then  $\mathcal{F}_\omega$  is compactifiable iff  $c(\omega) = b_1(M)$ , the first Betti number. In this case the cup-product  $\smile : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \rightarrow H^2(M, \mathbb{Z})$  is trivial.*

PROOF. Since  $\ker[\omega] = 0$ , we have  $\text{rk } \omega = b_1(M)$  and  $\mathcal{H} = H_\omega$ . Then the condition  $\text{rk } \mathcal{H} = \text{rk } \omega$  from Theorem 8 (iii) is equivalent to  $c(\omega) = b_1(M)$  and thus  $H_\omega = H_{n-1}(M)$ . Then  $H^1(M, \mathbb{Z}) = \langle z_i \rangle$ , where  $z_i$  are cocycles dual to  $[\gamma_i]$ , a basis of  $H_\omega$ , and  $\gamma_i \cap \gamma_j = \emptyset$  implies  $z_i \smile z_j = 0$ .  $\square$

So if  $\ker[\omega] = 0$  and  $\smile \neq 0$ , then  $\mathcal{F}_\omega$  has a minimal component. If, however,  $\smile \equiv 0$ , then both cases are possible. Indeed, on the one hand, in any cohomology class  $[\omega]$ ,  $\text{rk } \omega > 1$ , there exists a Morse form with minimal foliation [1]. On the other hand, the foliation can be compactifiable:

*Example 10.* Consider a connected sum  $M = \#_{i=1}^p (S^1 \times S^n)_i$ ,  $n > 1$ ; see Figure 2. Then  $b_1(M) = b'_1(M) = p$  (Example 1), which by (7) gives  $\smile \equiv 0$ . Consider  $\omega$  given on each  $(S^1 \times S^n)_i$  by  $\omega_i = \alpha_i dt$ , where  $t$  is a coordinate on  $S^1$  and all  $\alpha_i \in \mathbb{R}$  are independent over  $\mathbb{Q}$  so that  $\text{rk } \omega = p$ . Obviously,  $\mathcal{F}_\omega$  is compactifiable (its compact leaves are spheres  $S^n$ ).

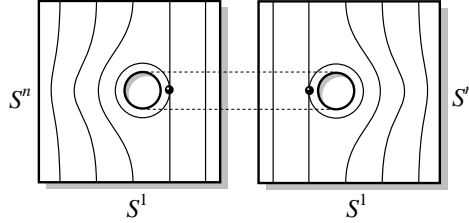


Figure 2. A foliation on a connected sum  $(S^1 \times S^n) \# (S^1 \times S^n)$ .

**Corollary 11.** For a two-dimensional genus  $g$  surface  $M_g^2$  it holds

$$\text{rk } \mathcal{H} \leq \text{rk } \omega \leq 2g - c(\omega) \leq 2g - \text{rk } \mathcal{H}. \quad (8)$$

If  $\text{rk } \mathcal{H} = g$ , then  $\mathcal{F}_\omega$  is compactifiable.

PROOF. The lower bound is by Theorem 8 (i). Since leaves are one-dimensional,  $\mathcal{H} \subseteq H_\omega \subseteq \ker[\omega]$  and  $\text{rk } \ker[\omega] = 2g - \text{rk } \omega$  gives the upper bound. If  $\text{rk } \mathcal{H} = g$  then (8) implies  $\text{rk } \omega = g$  and  $\mathcal{F}_\omega$  is compactifiable by Theorem 8 (iii).  $\square$

#### 4. Criterion for the presence of compact leaves

Farber *et al.* proved a necessary condition of existence of a compact leaf  $\gamma$  in terms of zero cup-product:

**Proposition 12** ([2, Proposition 9.14],[3, Proposition 3]). For so-called transitive Morse forms, if  $\mathcal{F}_\omega$  has a compact leaf with  $[\gamma] \neq 0$  then there exists a smooth closed 1-form  $\alpha$ ,  $0 \neq [\alpha] \in H^1(M, \mathbb{Z})$ , such that  $[\alpha] \smile [\omega] = 0$ .

The converse is, however, not true; see Counterexample 17 below. Moreover, no sufficient conditions for existing of a compact leaf can be given in cohomological terms: any cohomology class  $[\omega]$ ,  $\text{rk } \omega > 1$ , contains a form with minimal foliation [1].

We call 1-forms  $\alpha$  and  $\beta$  *collinear* if  $\alpha \wedge \beta = 0$ . Using the notion of collinearity instead of zero cup-product, we will generalize Proposition 12 to an arbitrary (not necessarily transitive) Morse form and refine it to a criterion. For closed 1-forms the equation  $\alpha \wedge \beta = 0$  implies  $[\alpha] \smile [\beta] = 0$  but not *vice versa*, so collinearity is a stronger condition.

Denote  $\text{Supp } \alpha = M \setminus \text{Sing } \alpha$ . If  $\alpha$  is closed, on  $\text{Supp } \alpha$  the integrable distribution  $\{\alpha = 0\}$  defines a foliation  $\mathcal{F}_\alpha$ .

**Lemma 13.** *For closed collinear 1-forms  $\alpha, \beta$ , on  $\text{Supp } \beta$  it holds  $\alpha = f(x)\beta$ , where  $f(x)$  is constant on leaves of  $\mathcal{F}_\beta$ . In particular, on  $\text{Supp } \alpha \cap \text{Supp } \beta$  it holds  $\mathcal{F}_\alpha = \mathcal{F}_\beta$ .*

PROOF. On  $\text{Supp } \beta$  there exists a smooth vector field  $\xi_x$  such that  $\beta(\xi_x) \neq 0$ . Consider  $f(x) = \frac{\alpha(\xi_x)}{\beta(\xi_x)}$ , which is well-defined: for any vector fields  $\xi_x, \eta_x$  we have  $\alpha(\xi_x)\beta(\eta_x) - \alpha(\eta_x)\beta(\xi_x) = (\alpha \wedge \beta)(\xi_x, \eta_x) = 0$ . Thus on  $\text{Supp } \beta$  we have  $\alpha = f(x)\beta$ .

Since  $\alpha$  and  $\beta$  are closed,  $df \wedge \beta = d\alpha - f d\beta = 0$ . Consider a vector field  $\xi$  tangent to the leaves of  $\mathcal{F}_\beta$  and  $\eta$  normal to the leaves. Then  $df \wedge \beta(\xi, \eta) = 0$  implies  $(df)(\xi) = 0$ , i.e.  $f$  is constant on leaves.  $\square$

**Proposition 14.** *Let  $\alpha$  be a smooth closed 1-form collinear with a Morse form  $\omega$ ;  $\alpha \neq 0$  and  $[\alpha] \in H^1(M, \mathbb{Z})$ . Then  $\text{Supp } \alpha$  is the union of a non-empty subset of compact leaves of  $\mathcal{F}_\omega$  and a subset of compactifiable leaves of  $\mathcal{F}_\omega$ . These leaves of  $\mathcal{F}_\omega$  are leaves of  $\mathcal{F}_\alpha$ .*

PROOF. All leaves of  $\mathcal{F}_\alpha$  are closed. Indeed, since  $[\alpha] \in H^1(M, \mathbb{Z})$ , it defines a smooth map  $F_{[\alpha]} : M \rightarrow S^1$ ,

$$F_{[\alpha]}(x) = e^{2\pi i \int_{x_0}^x \alpha}.$$

Obviously,  $F_{[\alpha]}$  is constant on leaves of  $\mathcal{F}_\alpha$  and the critical set of  $F_{[\alpha]}$  coincides with  $\text{Sing } \alpha$ . So on  $\text{Supp } \alpha$  the map is regular and by the implicit function theorem each leaf of  $\mathcal{F}_\alpha$  (which is a connected component of a level  $F_{[\alpha]}^{-1}(y)$ ,  $y \in S^1$ ) is a closed codimension-one submanifold of  $\text{Supp } \alpha$  (not necessarily closed in  $M$ ).

Next, if for a leaf  $\gamma \in \mathcal{F}_\omega$  it holds  $\gamma \cap \text{Supp } \alpha \neq \emptyset$  then  $\gamma \subseteq \text{Supp } \alpha$ . Indeed, suppose there exists  $x_0 \in \gamma \cap \text{Sing } \alpha$ . By Lemma 13, on  $\text{Supp } \omega$  it holds  $\alpha = f(x)\omega$ ,

where the function  $f(x)$  is constant on leaves. Since  $x_0 \in \text{Supp } \omega$ , we have  $f(x_0) = 0$  and so  $f|_\gamma = 0$ , which gives  $\gamma \cap \text{Supp } \alpha = \emptyset$ ; a contradiction.

Similarly, if for a leaf  $\gamma \in \mathcal{F}_\alpha$  it holds  $\gamma \cap \text{Sing } \omega \neq \emptyset$  then  $\gamma \subseteq \text{Sing } \omega$ . However, since  $\text{Sing } \omega$  consists of isolated points, such a leaf  $\gamma$  would be a point. This gives  $\text{Supp } \alpha \cap \text{Sing } \omega = \emptyset$  and thus  $\text{Supp } \alpha \subseteq \text{Supp } \omega$ .

Now Lemma 13 implies that all leaves of  $\mathcal{F}_\alpha$  are leaves of  $\mathcal{F}_\omega$ . Since all leaves of  $\mathcal{F}_\alpha$  are closed in  $\text{Supp } \alpha$ , the latter cannot contain any non-compactifiable leaves of  $\mathcal{F}_\omega$ . It cannot consist solely of non-compact compactifiable leaves of  $\mathcal{F}_\omega$  since their number is finite while  $\text{Supp } \alpha$  is open. Thus it must contain compact leaves of  $\mathcal{F}_\omega$ .  $\square$

**Lemma 15.** *In the conditions of Proposition 14, if  $[\alpha] \neq 0$  then  $\mathcal{F}_\alpha$  has a compact leaf with  $[\gamma] \neq 0$ .*

PROOF. Following the reasoning of [4] it is easy to show that (2) holds for  $\alpha$  even though it is not a Morse form. Since its  $\Delta$  consists of  $\text{Sing } \alpha$  and a finite number of compactifiable leaves,  $\text{rk } \alpha$  is determined by  $DH_\alpha$ . However, if  $[\gamma] = 0$  for any compact  $\gamma \in \mathcal{F}_\alpha$  then  $H_\alpha = 0$  and thus  $\text{rk } \alpha = 0$ , i.e.,  $[\alpha] = 0$ .  $\square$

Now we are ready to prove the main result of this section: a criterion for existence of a compact leaf.

**Theorem 16.** *The following conditions are equivalent:*

- (i)  $\mathcal{F}_\omega$  has a compact leaf  $\gamma$ ;
  - (ii) There exists a smooth function  $f(x) \not\equiv \text{const}$  such that  $df$  is collinear with  $\omega$ ;
  - (iii) There exists a smooth closed 1-form  $\alpha \neq 0$ ,  $[\alpha] \in H^1(M, \mathbb{Z})$ , collinear with  $\omega$ .
- Moreover,  $\gamma$  can be chosen with  $[\gamma] \neq 0$  iff  $\alpha$  can be chosen with  $[\alpha] \neq 0$ .

Note that  $f$  and  $\alpha$  are not required to be of Morse type.

PROOF. (i)  $\Rightarrow$  (ii), (iii): Let  $\gamma$  be a compact leaf. Consider a cylindrical neighborhood  $\mathcal{O}(\gamma) = \gamma \times I$  consisting of diffeomorphic leaves. Let  $(x^1, \dots, x^n)$  be local coordinates in  $\mathcal{O}(\gamma)$  such that  $(x^1, \dots, x^{n-1})$  are coordinates in  $\gamma$  and  $x^n$  in  $I$ . Consider a smooth function  $f(x) = f(x^n) \not\equiv \text{const}$  in  $\mathcal{O}(\gamma)$  and  $f(x) = 0$  on  $M \setminus \mathcal{O}(\gamma)$ . Let  $x \in \mathcal{O}(\gamma)$ ; consider the leaf  $\gamma' \ni x$ . Let  $\eta_1, \eta_2 \in T_x M$ ; then  $\eta_i = \xi_i + a_i n$ , where  $\xi_i \in T_x \gamma'$ ,  $a_i \in \mathbb{R}$ , and  $n \in T_x M \setminus T_x \gamma'$ . We obtain  $df(\eta_i) = a_i df(n)$  and  $\omega(\eta_i) = a_i \omega(n)$ . Thus  $df \wedge \omega(\eta_1, \eta_2) = 0$ , which proves (ii).

Consider now  $\alpha = f(x)\omega$ ; obviously,  $\alpha$  is closed and collinear with  $\omega$ . In addition, we can choose  $f$  such that  $[\alpha] \in H^1(M, \mathbb{Z})$ , which proves (iii). Finally, if  $[\gamma] \neq 0$  then there exists a cycle  $z \in H_1(M)$  such that  $z \cdot [\gamma] = 1$ ; choosing  $f$  non-negative we obtain  $\int_z \alpha \neq 0$ , thus  $[\alpha] \neq 0$ .

(ii), (iii)  $\Rightarrow$  (i): This has been shown as Proposition 14 and Lemma 15.  $\square$

Now Proposition 12 follows from Theorem 16. What is more, the same theorem shows that Proposition 12 is not a criterion:

*Counterexample 17.* The converse to Proposition 12 is not true for manifolds with  $b'_1(M) > 1$ ; see Section 2.3. Indeed, by (6) there exists a Morse form  $\omega$  on  $M$  such that  $c(\omega) = b'_1(M)$ . By Theorem 16 there exists a form  $\alpha$ ,  $0 \neq [\alpha] \in H^1(M, \mathbb{Z})$ , such that  $\alpha \wedge \omega = 0$  and thus  $[\alpha] \smile [\omega] = 0$ . The same foliation  $\mathcal{F}_\omega$  can be defined by a form of rank  $b'_1(M)$  [6, Theorem 4.1], so we can assume that  $\text{rk } \omega = b'_1(M) > 1$ . Then there exists a form  $\omega'$  with a minimal foliation and  $[\omega'] = [\omega]$  [1]; in particular,  $[\alpha] \smile [\omega'] = 0$ .

Recall that  $c(\omega) = \text{rk } H_\omega$  is the total number of homologically independent compact leaves of  $\mathcal{F}_\omega$ . Theorem 16 states that  $c(\omega) \neq 0$  iff there is a suitable  $[\alpha] \neq 0$ . This can be easily generalized to an arbitrary number  $k$ :  $c(\omega) \geq k$  iff there are  $k$  independent  $\alpha$ 's, which gives a criterion for existence of  $k$  homologically independent compact leaves:

**Theorem 18.** *The following conditions are equivalent:*

- (i)  $\mathcal{F}_\omega$  has  $k$  homologically independent compact leaves  $\gamma_i$ ;
- (ii) There exist  $k$  cohomologically independent smooth closed 1-forms  $\alpha_i$ ,  $[\alpha_i] \in H^1(M, \mathbb{Z})$ , collinear with  $\omega$ .

*If the above conditions hold for  $k = b'_1(M)$  then  $\mathcal{F}_\omega$  is compactifiable.*

PROOF. (i)  $\Rightarrow$  (ii): For each  $\gamma_i$  construct a form  $\alpha_i$ ,  $[\alpha_i] \neq 0$ , as in Theorem 16. Consider a Poincaré duality map  $D$  that satisfies (1) for  $\gamma_i$ . Since  $\int_{D\gamma_i} \alpha_j = \delta_{ij}$ , all  $[\alpha_i]$  are independent.

(ii)  $\Rightarrow$  (i): As has been noted in Lemma 15,  $\text{rk } \alpha_i$  is determined by  $DH_{\alpha_i}$ . By Proposition 14 we have  $H_{\alpha_i} \subseteq H_\omega$  and thus the rank of the whole system  $\langle [\alpha_1], \dots, [\alpha_k] \rangle$  is determined by  $H_\omega$ , which implies  $c(\omega) = \text{rk } H_\omega \geq k$ .

Finally, by (5),  $c(\omega) \geq k = b'_1(M)$  implies  $m(\omega) = 0$ , i.e.  $\mathcal{F}_\omega$  is compactifiable.  $\square$

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