Mathematical Notes, Vol. 60, No. 6, 1996

Properties of Morse Forms that Determine Compact Foliations on M_g^2

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KEY WORDS: two-dimensional manifold, foliation, Morse form, integration over cycles.

In [1, 2] P. Arnoux and G. Levitt showed that the topology of the foliation of a Morse form ω on a compact manifold is closely related to the structure of the integration mapping $[\omega]\colon H_1(M) \to \mathbb{R}$. In this paper we consider the foliation of a Morse form on a two-dimensional manifold M_g^2. We study the relationship of the subgroup $\operatorname{Ker}[\omega] \subset H_1(M_g^2)$ with the topology of the foliation. We consider the structure of the subgroup $\operatorname{Ker}[\omega]$ for a compact foliation and prove a criterion for the compactness of a foliation.

§1. Preliminary definitions

Consider a closed form ω with Morse singularities on M_g^2. This form determines a foliation $\mathcal{F}$ on $M_g^2 \setminus \operatorname{Sing}\omega$.

Let us define a foliation with singularities $\mathcal{F}_\omega$ on M_g^2 as follows.

Suppose that the foliation $\mathcal{F}$ is locally (in a sufficiently small neighborhood of a singular point $p \in \operatorname{Sing}\omega$) determined by the levels of a function f_p such that $f_p(p) = 0$.

Definition 1. A *nonsingular leaf of a foliation* $\mathcal{F}_\omega$ is a leaf $\gamma \in \mathcal{F}$ such that $\gamma \cap f_p^{-1}(0) = \varnothing$ for all $p \in \operatorname{Sing}\omega$.

Put $F_p = p \cup \{\gamma \in \mathcal{F} \mid \gamma \cap f_p^{-1}(0) \neq \varnothing\}$. Also put $F = \bigcup_{p \in \operatorname{Sing}\omega} F_p$.

Definition 2. A *singular leaf of a foliation* $\mathcal{F}_\omega$ is a connected component of F.

There is only a finite number of singular leaves (because the form is Morse).

A foliation $\mathcal{F}_\omega$ is called *compact* if all its leaves are compact.

A closed form ω determines the mapping $[\omega]\colon H_1(M_g^2) \to \mathbb{R}$ (integration over cycles). The image of this mapping $\operatorname{Im}[\omega]$ represents the period group of the form ω. Note that $\operatorname{rk}\operatorname{Im}[\omega] = \operatorname{dirr}\omega + 1$, where $\operatorname{dirr}\omega$ is the degree of irrationality of the form ω.

If $\operatorname{dirr}\omega \leq 0$, then the foliation $\mathcal{F}_\omega$ is compact [3]. If $\operatorname{dirr}\omega \geq g$, then the foliation $\mathcal{F}_\omega$ has a noncompact leaf [4]. If $0 < \operatorname{dirr}\omega < g$, then the foliation can be compact as well as noncompact. The study of the subgroup $\operatorname{Ker}[\omega]$ yields a condition for the compactness of a foliation in the latter case also.

Consider the intersection operation of 1-cycles

$$\varphi\colon H_1(M_g^2) \times H_1(M_g^2) \to \mathbb{Z}.$$

This operation is a nondegenerate skew-symmetric bilinear mapping.

By φ_ω denote the restriction of the mapping φ to the subgroup $\operatorname{Ker}[\omega] \subset H_1(M_g^2)$:

$$\varphi_\omega\colon \operatorname{Ker}[\omega] \times \operatorname{Ker}[\omega] \to \mathbb{Z}.$$

Obviously, $\operatorname{rk}\operatorname{Ker}\varphi_\omega \leq \operatorname{rk}\operatorname{Ker}[\omega] = 2g - (\operatorname{dirr}\omega + 1)$. For small values of $\operatorname{dirr}\omega$ a sharper estimate exists.

Proposition 1. $\operatorname{rk}\operatorname{Ker}\varphi_\omega \leq \operatorname{dirr}\omega + 1$.

Translated from *Matematicheskie Zametki*, Vol. 60, No. 6, pp. 942–945, December, 1996.
Original article submitted June 18, 1996.

0001-4346/96/6056-0714 $15.00

Proof. Suppose that $\operatorname{Ker}\varphi_\omega = \langle z_1, \dots, z_k \rangle$. By Dz_i denote the cycles dual to z_i; then $Dz_i \circ z_j = \delta_{ij}$. Assume that $\sum n_i \int_{Dz_i} \omega = 0$ and put $z = \sum n_i Dz_i$. We see that $z \in \operatorname{Ker}[\omega]$ and $z \circ z_j = n_j$. All the n_j equal zero because $z_j \in \operatorname{Ker}\varphi_\omega$. Thus, the integrals $\int_{Dz_i} \omega$ are linearly independent over $\mathbb{Q}$ and $\operatorname{dirr}\omega \geq k-1$. The proposition is proved. □

A subgroup $H \subset H_1(M_g^2)$ is called *isotropic with respect to cycle intersection* if $x \circ y = 0$ for all $x, y \in H$. An isotropic subgroup H is *maximal* if for all $x \notin H$ there exists a $y \in H$ such that $x \circ y \neq 0$. Note that the subgroup $\operatorname{Ker}\varphi_\omega$ is isotropic.

Proposition 2. *Let H_0 be a maximal isotropic subgroup in the group* $\operatorname{Ker}[\omega]$. *Then*

$$\operatorname{rk} H_0 = \frac{1}{2}(\operatorname{rk}\operatorname{Ker}[\omega] + \operatorname{rk}\operatorname{Ker}\varphi_\omega).$$

Proof. Since the mapping φ_ω is a symplectic form on $\operatorname{Ker}[\omega] \otimes \mathbb{R}$, the statement follows by splitting the symplectic space $\operatorname{Ker}[\omega] \otimes \mathbb{R}$ into an orthogonal direct sum of two-dimensional nondegenerate and one-dimensional degenerate subspaces. □

To each nonsingular leaf $\gamma \in \mathcal{F}_\omega$ assign its homology class $[\gamma]$. The image of all compact nonsingular leaves under this correspondence generates a subgroup in $H_1(M_g^2)$; denote it by H_ω. Note that the subgroup H_ω is isotropic and $H_\omega \subset \operatorname{Ker}[\omega]$.

§2. Properties of Morse forms that determine compact foliations

Theorem 1. *Suppose that a foliation on M_g^2 is determined by a Morse form ω. If the foliation $\mathcal{F}_\omega$ is compact, then*

$$\operatorname{rk}\operatorname{Ker}\varphi_\omega = \operatorname{dirr}\omega + 1.$$

Proof. Nonsingular compact leaves of $\mathcal{F}_\omega$ generate a subgroup $H_\omega \subset H_1(M_g^2)$. There is also a finite number of singular leaves, say, $\gamma_1^0, \dots, \gamma_k^0$. Consider the embeddings $j_s\colon \gamma_s^0 \to M_g^2$, $s = 1, \dots, k$, and the induced homology mappings $j_{s*}\colon H_1(\gamma_s^0) \to H_1(M_g^2)$. For each group $H_1(\gamma_s^0)$, choose its maximal subgroup Z_s so that the image $j_{s*}Z_s \subset H_1(M_g^2)$ is an isotropic subgroup.

Consider the subgroup $H_0 = \langle H_\omega, j_{s*}Z_s, s = 1, \dots, k \rangle$, which is obviously isotropic. Moreover, we have $H_0 \subset \operatorname{Ker}[\omega]$.

Consider a cycle $z \in H_1(M_g^2)$ such that $z \circ H_0 = 0$; then $z \circ H_\omega = 0$. In the paper [5] it was shown (see the proof of Theorem 1.2) that the cycle z can be realized by a curve $\alpha \subset \cup\gamma_s^0$ provided that the foliation $\mathcal{F}_\omega$ is compact and $z \circ H_\omega = 0$. Let us assume that $\alpha = \sum \alpha_s$, where $\alpha_s \subset \gamma_s^0$. We have $z \circ j_{s*}Z_s = 0$ for all s. Hence, $[\alpha_s] \circ j_{s*}Z_s = 0$ because $\alpha_p \cap \gamma_0^s = \varnothing$ whenever $p \neq s$. Since $\alpha_s \subset \gamma_0^s$, we get $[\alpha_s] \in \operatorname{Im} j_{s*}$. Thus, $[\alpha_s] \in j_{s*}Z_s$ and $z \in H_0$ (by the construction of the group Z_s).

So H_0 is a maximal subgroup of $H_1(M_g^2)$. Thus, $\operatorname{rk} H_0 = g$ (this is shown in [5]). On the other hand, H_0 is a maximal subgroup of $\operatorname{Ker}[\omega]$, and Proposition 2 implies that

$$\operatorname{rk} H_0 = \frac{1}{2}(\operatorname{rk}\operatorname{Ker}[\omega] + \operatorname{rk}\operatorname{Ker}\varphi_\omega).$$

Since $\operatorname{dirr}\omega = 2g - 1 - \operatorname{rk}\operatorname{Ker}[\omega]$, the theorem is proved. □

Remark. The converse statement is not true, i.e., there exist noncompact foliations such that the previous relation holds.

§3. A criterion for the compactness of a foliation

In [4] it is shown that if there are g compact leaves that are homologically independent, then all the leaves are compact. Taking into account the structure of the subgroup $\operatorname{Ker}[\omega]$, let us strengthen this criterion of compactness. Let us consider the intersection $H_\omega \cap \operatorname{Ker}\varphi_\omega$.

Theorem 2. *A foliation $\mathcal{F}_\omega$ is compact if and only if*

$$\operatorname{rk}(H_\omega \cap \operatorname{Ker}\varphi_\omega) \geq \operatorname{dirr}\omega.$$

Proof. If $\operatorname{dirr}\omega \leq 0$, then the compactness of the foliation follows from [3]. Assume that $\operatorname{dirr}\omega \geq 1$, then $k = \operatorname{rk}(H_\omega \cap \operatorname{Ker}\varphi_\omega) \geq 1$.

Consider the nonsingular leaves $\gamma_i \in \mathcal{F}_\omega$ such that $H_\omega \cap \operatorname{Ker}\varphi_\omega = \langle [\gamma_1], \dots, [\gamma_k] \rangle$. Proposition 1 implies that the integrals $\int_{D[\gamma_i]} \omega$ are linearly independent over $\mathbb{Q}$, and $\operatorname{dirr}\omega \geq k-1$. By assumption, $\operatorname{dirr}\omega \leq k$; thus $\operatorname{dirr}\omega$ is determined by means of integrals over $D[\gamma_1], \dots, D[\gamma_k]$ and, possibly, over some cycle z.

Let us cut M_g^2 along the fibers γ_i. Since they are homologically independent, we get a connected manifold M' with boundary. The number of connected components of the boundary is $2k$. By ω' denote the restriction of the form ω to M'. Since the cycles $D[\gamma_i]$ vanish after cutting, we have $\operatorname{dirr}\omega' \leq 0$.

Attaching a disk D_i^2 to each connected component of the boundary, we obtain a manifold M_{g-k}^2. Extend the form ω' to the disks D_i^2 in such a way that the form ω'' obtained on every D_i^2 is a Morse form and has exactly one singular point, either minimum-like or maximum-like. Obviously, $\operatorname{dirr}\omega'' \leq 0$ and the foliation $\mathcal{F}_{\omega''}$ is compact. So the foliation $\mathcal{F}_\omega$ is also compact. One implication of the theorem is proved.

Conversely, assume that the foliation $\mathcal{F}_\omega$ is compact. Let us show that $\operatorname{Im}[\omega] = D\operatorname{Ker}\varphi_\omega$. Indeed, it follows from the proof of Proposition 1 that $D\operatorname{Ker}\varphi_\omega \subseteq \operatorname{Im}[\omega]$. Theorem 1 implies that $\operatorname{rk}\operatorname{Ker}\varphi_\omega = \operatorname{dirr}\omega + 1$. On the other hand, $\operatorname{rk}\operatorname{Im}[\omega] = \operatorname{dirr}\omega + 1$, consequently, $\operatorname{Im}[\omega] = D\operatorname{Ker}\varphi_\omega$. So, if the foliation $\mathcal{F}_\omega$ is compact, then $H_1(M_g^2) = \operatorname{Ker}[\omega] \oplus D\operatorname{Ker}\varphi_\omega$.

Suppose that $H_\omega = \langle [\gamma_1], \dots, [\gamma_N] \rangle$. The proof of Theorem 2.2 of the paper [5] implies that

$$\operatorname{dirr}\omega + 1 = \operatorname{rk}_{\mathbb{Q}}\left\{ \int_{D\gamma_1} \omega, \dots, \int_{D\gamma_N} \omega \right\}.$$

Let us reindex the leaves in such a way that $\int_{D\gamma_i} \omega \neq 0$ for all $i \leq s$ and $\int_{D\gamma_i} \omega = 0$ for all $i > s$. Then

$$\operatorname{dirr}\omega + 1 = \operatorname{rk}_{\mathbb{Q}}\{\textstyle\int_{D\gamma_1} \omega, \dots, \int_{D\gamma_s} \omega\} \leq \operatorname{rk}\langle [\gamma_1], \dots, [\gamma_s] \rangle \leq \operatorname{rk}(H_\omega \cap \operatorname{Ker}\varphi_w)$$

because $[\gamma_i] \in \operatorname{Ker}\varphi_\omega$ whenever $\int_{D\gamma_i} \omega \neq 0$, as it is shown above. The theorem is proved. □

So the following condition is sufficient for the compactness of a foliation: there exist $\operatorname{dirr}\omega$ homologically independent leaves such that their homology classes belong to the subgroup $\operatorname{Ker}\varphi_\omega$. In particular, if $\operatorname{dirr}\omega = 1$ and there exists a nonhomological to zero leaf γ such that $[\gamma] \in \operatorname{Ker}\varphi_\omega$, then the foliation is compact.

I wish to express my gratitude to Professor A. S. Mishchenko for his interest in my work and for useful discussions.

This research was partially supported by the Russian Foundation for Basic Research under grant No. 96-01-00276.

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Translated by S. S. Anisov