Ranks of collinear Morse forms

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Abstract

On a smooth closed n-manifold, we consider Morse forms with wedge-product zero; we call such forms collinear. This is an equivalence relation. Collinearity classes are classified by the underlying foliation; so, in other words, we study the set of Morse forms that define the same foliation. We describe the set of the ranks of such forms and show how it is related to the structure of the foliation and the manifold.

Key words: Collinear Morse forms, Morse form foliation, form's rank, minimal components, compact leaves
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1 Introduction and statement of main results

Let M be a connected smooth closed oriented n-dimensional manifold. A Morse form on M is a closed 1-form with Morse type singularities—locally the differential of a Morse function. We study Morse forms ω, ω' on M with wedge-product zero, $\omega \wedge \omega' = 0$; we call such forms *collinear*.

Morse forms are "typical" among closed 1-forms: the set of Morse forms is open and dense in the space of all closed 1-forms [1]. By the Morse lemma, near its critical points a Morse function has quadratic structure. Since Morse forms are typical, this explains the ubiquity of quadratic forms and functions in physics and life.

On the other hand, collinear 1-forms appear in many problems of theoretical physics, for example, in general relativity and quantum cosmology. Collinear 1-forms which are the Weyl tensor invariants arise in the problem of classification of type I vacuum solutions with aligned Papapetrou fields [2]. The triplet

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ansatz, which figures in various problems of metric–affine theory of gravity [3, 4], defines three collinear 1-forms.

Collinearity is a reflexive and symmetric relation; we show that for Morse forms—in fact, for forms with small enough set of singularities—it is also transitive (Corollary 3.3). We denote the equivalence class of ω by $[\omega]$.

The corresponding foliation $\mathcal{F}_\omega$ is a characteristic of the class (Lemma 3.2). In geometric terms, much of our paper can be thought of as the inverse problem: *the study of the forms defining a given foliation* $\mathcal{F}$ as a geometric object regardless of the form—in physics it is often observable experimentally. Invariants of $[\omega]$ are invariants of $\mathcal{F}$. Particularly important for the theory of foliations is that to study the properties of a given $\mathcal{F}$ one can choose any $\omega' \in [\omega]$ that best fits the needs of the problem at hand.

In algebraic terms, two Morse forms are collinear iff $\omega' = f(x)\omega$, where $f(x)$ is a non-vanishing smooth function such that $df \wedge \omega = 0$, i.e., constant on leaves of $\mathcal{F}_\omega$ (Lemma 3.6); we call such functions *admissible* for ω. Obviously, admissibility of a function is an invariant of a class and the set of admissible functions is its characteristic. The set of functions admissible for a given class $[\omega]$ has rich algebraic structure (Proposition 3.9). In summary, a class of collinear Morse forms can be described as

$$\begin{aligned}[\omega] &\equiv \{\omega' \mid \omega' \wedge \omega = 0\} \\ &= \{\omega' \mid \mathcal{F}_{\omega'} = \mathcal{F}_\omega\} \\ &= \{\omega' \mid \omega' = f(x)\omega;\ f \text{ is admissible}\}.\end{aligned}$$

The foliation $\mathcal{F}$ of $[\omega]$ defines the so-called foliation graph Γ (Section 2.3), which is, obviously, an invariant of the class. The foliation digraphs $\overrightarrow{\Gamma}$ (oriented along the form gradient) of different forms in $[\omega]$ coincide up to global inversion of the orientation (Corollary 3.7). Thus, a foliation as a geometric object, regardless of the 1-form, defines the orientation $\pm\overrightarrow{\Gamma}$ on its foliation graph Γ, in particular, the presence of (directed) circuits in it.

For a collinearity class $[\omega]$ of Morse forms, we study the set of ranks

$$\mathcal{R} = \{\operatorname{rk}\omega' \mid \omega' \in [\omega]\},$$

where the rank of a form is the rank of its group of periods:

$$\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}}\left\{\textstyle\int_z \omega \mid z \in H_1(M)\right\}.$$

Note again that $\mathcal{R}$ is an invariant of the foliation $\mathcal{F}$ as a geometric object. We calculate its maximum and estimate its minimum and the range $\max(\mathcal{R}) - \min(\mathcal{R})$ in terms of the homological structure of the set of compact leaves, the local groups of periods P_j roughly corresponding to minimal components (Proposition 4.2), and the presence of circuits of the foliation digraph $\overrightarrow{\Gamma}_\omega$ (Proposition 4.8). Our main theorem (Theorem 4.11) summarizes these results. The set $\mathcal{R}$ is not necessarily a segment $[\min(\mathcal{R}), \max(\mathcal{R})]$: it may have gaps; this depends on the algebraic structure of P_j (Proposition 4.5).

Properties of $\mathcal{F}$ are closely connected with $\mathcal{R}$. We show (Corollary 5.1) that $\min(\mathcal{R}) \leq 1$ iff $\mathcal{F}$ is compactifiable (has no minimal components) and if $1 \in \mathcal{R}$, then $\mathcal{F}$ has homologically non-trivial compact leaves; on the other hand, if $\max(\mathcal{R}) > b_1'(M)$, then $\mathcal{F}$ has minimal components, where $b_1'(M)$ is the maximum rank of a (non-Abelian) free quotient group of the fundamental group $\pi_1(M)$. Then we classify the cases when $|\mathcal{R}| = 1$ in terms of the foliation structure (Corollary 5.2).

The value $b_1'(M)$, introduced by Arnoux and Levitt [5], plays important role in the study of foliations. Unfortunately, we are not aware of any practical ways of calculating $b_1'(M)$ for a specific manifold. We show, however, that $b_1'(M) \leq h(M)$, where $h(M)$ is the maximal rank of a subgroup of $H^1(M)$ with zero cup-product (Proposition 4.10); unlike $b_1'(M)$, it can be nicely calculated for many specific manifolds [6, 7]. This gives a weaker but more practical upper bound whenever $b_1'(M)$ is involved.

Especially interesting is the case when the first Betti number $b_1(M) \in \mathcal{R}$, i.e., $\max(\mathcal{R}) = b_1(M)$, which is the maximum value of $\operatorname{rk}\omega$ possible for a given manifold M. This gives information not only on $\mathcal{F}$ but also on the topology of M, namely, the structure of its cup-product $\smile$: in this case $\operatorname{rk}\ker\smile\, \geq c(\omega)$, the number of homologically independent compact leaves of $\mathcal{F}$ (Theorem 5.3); if in addition $\mathcal{F}$ is compactifiable then $c(\omega) = b_1(M)$ and $\smile\, \equiv 0$ (Corollary 5.4); if $b_1(M) \neq 0$ and $\smile$ is non-degenerate then $c(\omega) = 0$ and $\mathcal{F}$ has minimal components (Corollary 5.5).

Note that the condition $b_1(M) \in \mathcal{R}$ for the foliation $\mathcal{F}_\omega$ of a given form ω can be met even if $\operatorname{rk}\omega$ is small (Example 5.7): e.g., it is enough that $\sum \operatorname{rk} P_j = b_1(M)$ for the above-mentioned P_j (Corollary 5.6). This illustrates how by studying the foliation of a given form ω one can predict the existence of another form $\omega' \in [\omega]$ that gives better information on $\mathcal{F}_\omega$ and even on M: here, a form ω' with $\operatorname{rk}\omega' = b_1(M)$.

The paper is organized as follows. In Section 2, we give some necessary definitions and facts concerning Morse form foliations. In Section 3, we study collinear Morse forms and in Section 4, the set $\mathcal{R}$ of their ranks. Finally, in Section 5 we connect our findings with the manifold and foliation structure.

2 Morse form foliation

Let us introduce for future reference some useful notions and facts about Morse forms and their foliations.

Let M be a connected smooth closed oriented n-dimensional manifold. A closed 1-form ω on M is called a *Morse form* if it is locally the differential of a Morse function. Denote by $\operatorname{Sing}\omega = M \setminus \operatorname{Supp}\omega$ the set of its singularities. This set is finite since the singularities are isolated and M is compact.

In the sequel we shall only consider Morse forms unless otherwise stated.

2.1 Leaves

On $\operatorname{Supp}\omega$ the form ω defines a foliation $\mathcal{F}_\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is called *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact (note that compact leaves are compactifiable); otherwise it is called *non-compactifiable.* If a foliation contains only compactifiable leaves, then it is called *compactifiable.*

Consider the group $H_\omega \subseteq H_{n-1}(M)$ generated by the homology classes of all compact leaves. Since M is closed and oriented, H_ω is finitely generated and free; it has a basis consisting of homology classes of leaves:

$$H_\omega = \langle [\gamma_1], \dots, [\gamma_{c(\omega)}] \rangle, \tag{1}$$

$\gamma_i \in \mathcal{F}_\omega$ [6]. The value $c(\omega) = \operatorname{rk} H_\omega$ is the number of homologically independent compact leaves.

While $\mathcal{F}_\omega$ is defined only on $\operatorname{Supp}\omega$, we can extend it to the whole M as a *singular foliation* $\overline{\mathcal{F}}_\omega$:

Definition 2.1 ([8]) *A* singular foliation $\overline{\mathcal{F}}_\omega$ *is a decomposition of M into* leaves*: two points $p, q \in M$ belong to the same leaf if there exists a path $\alpha : [0,1] \to M$, $\alpha(0) = p$, $\alpha(1) = q$, with $\omega(\dot\alpha(t)) = 0$ for all t.*

A *singular* leaf of $\overline{\mathcal{F}}_\omega$ contains a singularity. A 1-form is called *generic* if each its singular leaf contains precisely one singularity [8, Definition 9.1]. The set of generic Morse forms is dense in the space of all 1-forms [8, Lemma 9.2] [9], though it is not necessarily open.

Leaves compactified by one singularity are only found next to compact leaves:

Lemma 2.2 *Let $\gamma^0 \in \mathcal{F}_\omega$ be a non-compact compactifiable leaf such that $\gamma^0 \cup s$*

is compact for some $s \in \operatorname{Sing}\omega$*. Then in any neighborhood of* $\overline{\gamma^0} = \gamma^0 \cup s$ *there exists a compact leaf* $\gamma \in \mathcal{F}_\omega$*.*

PROOF. For a two-dimensional surface M_g^2 the fact has been proved in [10] (using similar considerations), so assume $\dim M \geq 3$.

Consider a small cylindrical neighborhood U of $\overline{\gamma^0}$ such that $U \cap \operatorname{Sing}\omega = \{s\}$. In this neighborhood $\omega = df$; assume $f(\gamma^0) = 0$. The set $U \setminus \overline{\gamma^0}$ has two connected components U_1, U_2; see Fig. 1.

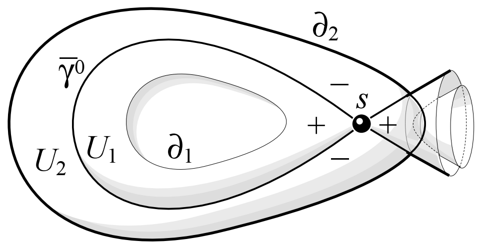

Fig. 1. A neighborhood U of $\overline{\gamma^0}$, a non-compact leaf compactified by a singularity s, $\operatorname{ind} s = 1$. The function f, $\omega = df$, has constant sign in U_1.

By the Morse lemma, in a neighborhood of s there are local coordinates $x_1, \ldots, x_n$, $n = \dim M$, such that $x_i(s) = 0$ and $f(x) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2$, $k = \operatorname{ind} s$. In a neighborhood of a singularity of index k and $n - k$ the foliation has the same topological structure.

If $\operatorname{ind} s = 1$ or $n - 1$, then the singular level $\{f = 0\}$ is locally a cone and thus $\{f = 0\} \setminus s$ is not connected—the case shown in Fig. 1; non-singular levels near s are one-sheeted and two-sheeted hyperboloids. For any other $\operatorname{ind} s$ the set $\{f = 0\} \setminus s$ is connected; nearby non-singular levels are one-sheeted hyperboloids.

Therefore locally there are at most two (non-compact) leaves adjoining s, at least one of which is a part of γ^0. Thus at least one of the two components of U, say U_1, does not intersect (locally) with the singular leaf containing γ^0. In particular, f does not change its sign in U_1; assume $f|_{U_1} > 0$. We can even assume $f|_{\partial_1} > 0$, where ∂_1 is the connected component of ∂U corresponding to U_1. Since ∂_1 is compact, $\varepsilon = \min_{\partial_1} f(x) > 0$. Therefore $f^{-1}(\frac{\varepsilon}{2}) \subset U_1$. This non-empty compact set is a leaf of $\mathcal{F}_\omega$. $\square$

2.2 Decomposition of the manifold

A foliation $\mathcal{F}_\omega$ defines a complex-like decomposition of M into a finite number of mutually disjoint sets [6]:

$$M = \Big(\bigcup C_i^{max}\Big) \cup \underbrace{\Big(\bigcup C_j^{min}\Big) \cup \Big(\bigcup \gamma_k^0\Big) \cup \operatorname{Sing}\omega}_{\Delta}. \qquad (2)$$

A *maximal component* C_i^{max} of the foliation is a connected component of the union of all compact leaves. If $\operatorname{Sing}\omega \neq \emptyset$, each maximal component is a cylinder over a compact leaf:

$$C_i^{max} \cong \gamma_i \times (0,1), \tag{3}$$

where the diffeomorphism maps γ_i to leaves of $\mathcal{F}_\omega$. The number of maximal components is finite.

A *minimal component* C_j^{min} is a connected component of the union of all non-compactifiable leaves. The number $m(\omega)$ of minimal components is finite. Each non-compactifiable leaf is dense in its minimal component [11].

Non-compact compactifiable leaves γ_k^0 and singularities are the boundaries of components C_i^{max} and C_j^{min}, which are open. The number of non-compact compactifiable leaves and singularities is also finite.

In homology terms, (2) takes the following form:

Lemma 2.3 *Let $\{\gamma_i\} \subset \mathcal{F}_\omega$ be compact leaves such that $[\gamma_i]$ form a basis of H_ω. Then*

$$H_1(M) = DH_\omega \oplus i_* H_1(\Delta), \tag{4}$$

where $i : \Delta \hookrightarrow M$ is the inclusion and $D : H_{n-1}(M) \to H_1(M)$ is a Poincaré duality map such that the intersection product $D[\gamma_i] \cdot [\gamma_j] = \delta_{ij}$.

This refines the result from [6] [1],

$$H_1(M) = \langle DH_\omega, i_* H_1(\Delta)\rangle,$$

in noting that if $z \in DH_\omega \cap i_* H_1(\Delta)$, then $z = \sum n_i D[\gamma_i]$ and $z \cdot [\gamma_i] = 0$ for all i since $\gamma_i \cap \Delta = \emptyset$; thus all $n_i = 0$.

2.3 Foliation graph

The configuration formed by maximal components in the decomposition (2) is described by the *foliation graph* Γ_ω. Rewrite (2) as

$$M = \left(\bigcup C_i^{max}\right) \cup \left(\bigcup \Delta_j\right), \tag{5}$$

where Δ_j are connected components of the union Δ of all non-compact leaves and singularities.

[1] We use the notation $\langle A, B\rangle$ for the module generated by A and B.

By (3), $\partial\mathcal{C}_i^{max} \subseteq \Delta$ consists of one or two connected components; thus each $\mathcal{C}_i^{max}$ adjoins one or two of Δ_j. This allows representing M as a connected graph (allowing loops and multiple edges) Γ_ω with vertices Δ_j and edges $\mathcal{C}_i^{max}$: an edge $\mathcal{C}_i^{max}$ is incident to a vertex Δ_j if $\partial\mathcal{C}_i^{max} \cap \Delta_j \neq \emptyset$; see Fig. 2. The graph can be directed by the increase of the local gradient defined by ω; we denote this digraph by $\overrightarrow{\Gamma}_\omega$.

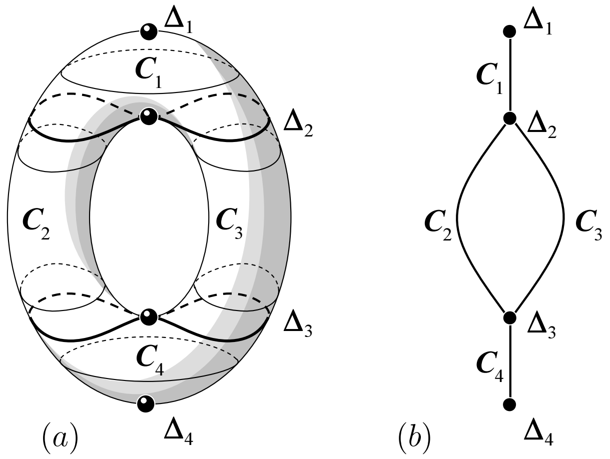

Fig. 2. (a) A foliation on a torus T^2 with four maximal components $\mathcal{C}_i$ and four sets Δ_j from (5)—of them, two singular leaves and two isolated singularities; (b) the corresponding foliation graph.

If $\operatorname{Sing}\omega = \emptyset$, then M is a fiber bundle over S^1 [12], with either $M = \mathcal{C}^{min}$ or $M = \mathcal{C}^{max}$. In the latter case, we assume the "graph" Γ_ω, suitably generalized, to have one loop edge and no vertices.

A *semi-circuit* in a digraph is a cycle in the corresponding undirected graph; we choose an arbitrary orientation in it. If all edges of a semi-circuit go the same direction, it is called a *circuit*. A digraph is *acyclic* if it has no circuits; note that it can have semi-circuits, i.e., the corresponding undirected graph can have cycles. The following technical lemma is used in the proof of Proposition 4.8:

Lemma 2.4 *Let* $\overrightarrow{\Gamma}$ *be a directed acyclic graph (allowing loops and multiple edges). Then there exists a function* $f(x_i) > 0$ *on its edges such that for any semi-circuit* s *it holds* $\sum_s \pm f(x_i) = 0$*, where the sign is positive iff* x_i *is directed along the orientation in* s.

Indeed, for an edge $x = \overrightarrow{uv}$ we can choose $f(x) = t(v) - t(u)$, where $t(*)$ is the number of the vertex in a topological ordering of $\overrightarrow{\Gamma}$.

3 Collinear Morse forms

Definition 3.1 *1-forms ω, ω' such that $\omega \wedge \omega' = 0$ are called* collinear.

On the set of Morse forms (unlike arbitrary 1-forms) collinearity is an equivalence relation, and the foliation is its class characteristic. Indeed:

Lemma 3.2 *Let ω, ω' be Morse forms. Then $\omega \wedge \omega' = 0$ iff $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$.*

PROOF. Let $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$. Since on $\operatorname{Sing}\omega' = \operatorname{Sing}\omega$ it holds that $\omega \wedge \omega' = 0$, consider $x \notin \operatorname{Sing}\omega$, i.e., $x \in \gamma \in \mathcal{F}_\omega$. Choose an $n \in T_xM \setminus T_x\gamma$. Since projecting T_xM on $\langle n\rangle$ along $T_x\gamma$ preserves both ω and ω', at x we also have $\omega \wedge \omega' = 0$.

Let now $\omega \wedge \omega' = 0$. Consider a path $\alpha : [0,1] \to M$ in a leaf $\gamma \in \mathcal{F}_\omega$, so $\omega(\dot\alpha) = 0$. Consider $x \in \alpha \setminus \operatorname{Sing}\omega$, $\xi \in T_x\alpha$, and $n \in T_xM \setminus T_x\gamma$; the equation $\omega \wedge \omega'(\xi, n) = 0$ implies $\omega'(\xi) = 0$. Thus $\omega'(\dot\alpha) = 0$ on $\alpha \setminus \operatorname{Sing}\omega$ and, by continuity, on the whole α, since $\operatorname{Sing}\omega$ is a finite. Similarly, $\omega'(\dot\alpha) = 0$ implies $\omega(\dot\alpha) = 0$. Therefore, the two forms define the same equivalence relation from Definition 2.1, i.e. $\overline{\mathcal{F}}_\omega = \overline{\mathcal{F}}_{\omega'}$ and thus $\mathcal{F}_\omega = \mathcal{F}_{\omega'}$. $\square$

Corollary 3.3 *For collinear Morse forms ω, ω' it holds that* $\operatorname{Sing}\omega' = \operatorname{Sing}\omega$.

Corollary 3.4 *For Morse forms, collinearity is an equivalence relation.*

Denote by $[\omega]$ the equivalence class of a Morse form ω or, in other words, the class of Morse forms that define a given Morse form foliation $\mathcal{F}_\omega$:

$$\begin{aligned}[\omega] &\equiv \{\omega' \in \text{Morse forms} \mid \omega' \wedge \omega = 0\}\\ &= \{\omega' \in \text{Morse forms} \mid \mathcal{F}_{\omega'} = \mathcal{F}_\omega\}.\end{aligned}$$

We shall study the structure of the forms that constitute $[\omega]$.

Proposition 3.5 *Let ω, ω' be Morse forms. Then $\omega' \in [\omega]$ iff $\omega' = f(x)\omega$ for some smooth non-vanishing function $f(x)$ on M such that $df \wedge \omega = 0$.*

PROOF. Let $\omega' = f(x)\omega$. Then it has Morse singularities and $d\omega' = 0$; thus $\omega \wedge \omega' = \omega \wedge f\omega = 0$, i.e., $\omega' \in [\omega]$.

Let now $\omega' \in [\omega]$. By Corollary 3.3 we have $\operatorname{Sing}\omega' = \operatorname{Sing}\omega$. On $M \setminus \operatorname{Sing}\omega$ consider a smooth vector field ξ such that $\omega(\xi_x) \neq 0$ for all x. The ratio

$$f(x) = \frac{\omega'(\xi_x)}{\omega(\xi_x)} \tag{6}$$

is a smooth non-vanishing function, which is well-defined since for any other such field η the collinearity condition gives

$$(\omega \wedge \omega')(\xi_x, \eta_x) = \omega(\xi_x)\omega'(\eta_x) - \omega(\eta_x)\omega'(\xi_x) = 0;$$

thus $\omega' = f(x)\omega$. Since $d\omega' = 0$, we obtain $df \wedge \omega = 0$. It remains to show that $f(x)$ can be smoothly continued to $\operatorname{Sing}\omega$ preserving these properties.

By the Morse lemma, in a neighborhood of $s \in \operatorname{Sing}\omega$ there exist coordinates x^i such that $x^i(s) = 0$ and $\omega(\xi) = \sum \pm x^i \xi^i$. Rewrite (6) as

$$f(x) = \frac{\sum a_{ij}(x) x^j \xi^i}{\sum \pm x^i \xi^i}, \tag{7}$$

where $a_{ij}(x) = \int_0^1 \frac{\partial \omega_i'(tx)}{\partial x^j} dt$. Collinearity of ω and ω' implies that $a_{ij}(x) = 0$ for $i \neq j$. Then for $x \neq 0$, (7) can be rewritten as

$$\sum a_{ii}(x) x^i \xi^i = \sum \pm f(x) x^i \xi^i,$$

which gives $f(x) = \pm a_{11}(x)$. It can be smoothly continued to $x = 0$. Since $a_{ij}(0) = \frac{\partial \omega_i'(0)}{\partial x^j}$ and ω' is a Morse form, the matrix $(a_{ij}(0))$ is non-degenerate; thus $f(0) = \pm a_{11}(0) \neq 0$. □

Lemma 3.6 *$df \wedge \omega = 0$ iff f is constant on leaves of $\mathcal{F}_\omega$.*

This can be proved by direct calculation.

Corollary 3.7 *If $\omega' \in [\omega]$, then $\overrightarrow{\Gamma}_{\omega'} = \pm \overrightarrow{\Gamma}_\omega$, i.e. the directed foliation graphs of collinear Morse forms either coincide or have opposite orientations.*

We have shown that $[\omega] = \{f(x)\omega \mid f(x) \neq 0 \text{ and } df \wedge \omega = 0\}$, so the study of $[\omega]$ reduces to the study of this class of functions.

Definition 3.8 *A smooth function $f(x) \neq 0$ such that $df \wedge \omega = 0$ is called* admissible *for the Morse form ω.*

By Proposition 3.5, a function admissible for ω is admissible for all forms in $[\omega]$. What is more, the set of admissible functions is a characteristic of the class $[\omega]$.

The algebraic structure of the set of functions admissible for ω is very rich:

Proposition 3.9 *Let $\varphi(y_1, \dots, y_m) \neq 0$ be a smooth function on $(\mathbb{R} \setminus \{0\})^m$ and $f_1(x), \dots, f_m(x)$ be admissible functions. Then $\varphi(f_1(x), \dots, f_m(x))$ is also admissible. In particular, if $f(x), g(x)$ are admissible, then so are $-f(x)$, $\frac{1}{f(x)}$,*

$f(x)g(x)$, and $f(x)^{g(x)}$; if $\operatorname{sgn} f(x) = \operatorname{sgn} g(x)$ or otherwise $f(x) \neq -g(x)$ for all x, then $f(x) + g(x)$ is admissible.

Proposition 3.10 *An admissible function $f(x)$ on M can be viewed as a function on the foliation graph Γ_ω as a 1-complex.*

PROOF. Consider the decomposition (5). By definition, an admissible function $f(x)$ is constant on leaves of $\mathcal{F}_\omega$. Since a leaf is dense in a minimal component, $f(x)$ is constant in each vertex Δ_i, and in maximal components C_i^{max} it can be thought of as defined on an edge of Γ_ω as an interval. □

4 Ranks of collinear Morse forms

Recall that $\operatorname{rk}\omega$ is the rank of its group of periods over $\mathbb{Q}$. In this section, we study the set of ranks of collinear forms

$$\mathcal{R} = \{\operatorname{rk}\omega' \mid \omega' \in [\omega]\}.$$

For non-singular forms, $|\mathcal{R}| = 1$; namely, $\mathcal{R} = \{r\}$, with $r = 1$ for compact foliations and $r \geq 2$ for minimal foliations; cf. Corollary 5.2.

As shown in Lemma 2.3,

$$H_1(M) = i_* H_1(\Delta) \oplus DH_\omega. \tag{8}$$

We shall construct the set $\mathcal{R}$ of two sets, $\mathcal{R}_\Delta$ and $\mathcal{R}_H$, that correspond to the first and the second summand.

4.1 The set $\mathcal{R}_\Delta$

Recall that Δ_j are connected components of Δ from (2) (cf. (5)):

$$\Delta = \left(\bigcup \mathcal{C}_l^{min}\right) \cup \left(\bigcup \gamma_i^0\right) \cup \operatorname{Sing}\omega.$$

If $\Delta_j \subseteq (\bigcup \gamma_i^0) \cup \operatorname{Sing}\omega$, then $\omega|_{\Delta_j} = 0$ and by Lemma 3.2 for $\omega' \in [\omega]$ also $\omega'|_{\Delta_j} = 0$. So the periods of ω' in Δ are defined only by those Δ_j that contain minimal components. In the sequel we shall consider only such Δ_j; denote by k their number:

$$k = |\{\Delta_j \subseteq \Delta \mid \Delta_j \cap \bigcup \mathcal{C}_i^{min} \neq 0\}|. \tag{9}$$

Obviously, $k \leq m(\omega)$, where $m(\omega)$ is the number of minimal components. Fig. 3 shows an example of strict inequality: a double torus as a connected sum $M_2^2 = T^2 \sharp T^2$ with a separate irrational winding on each T^2 and without any maximal components. While it has two minimal components, its foliation graph consists of the only vertex $\Delta_1 = \Delta$ and no edges. Note that this Morse form is not generic.

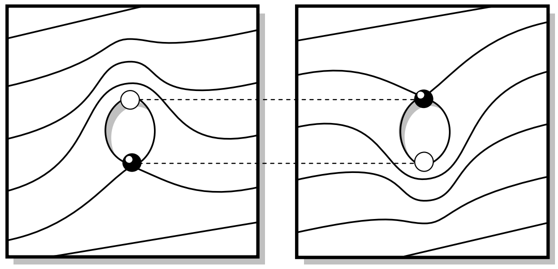

Fig. 3. Two tori T^2 with a hole, each one with its own irrational winding and a compactifiable leaf along the border of the hole. They are glued together by the holes into a double torus M_2^2 (as in Fig. 4, but the tube has zero length) with $m(\omega) = 2$, but $k = 1$. The singular leaf contains two singularities.

Lemma 4.1 *If ω is a generic Morse form, then $k = m(\omega)$.*

PROOF. Recall that each singular leaf of a generic form contains only one singularity.

Consider a connected component $\partial \subset \partial\mathcal{C}^{min}$. It is a part of a non-compact singular leaf γ, $\gamma \subset \overline{\mathcal{C}^{min}} \subset \Delta_j$; more specifically, ∂ is a compactifiable leaf compactified by one singularity. By Lemma 2.2, there exists a compact leaf close to ∂. Thus what is attached to $\mathcal{C}^{min}$ by ∂ is a maximal component. Therefore each Δ_j contains at most one minimal component. $\square$

Denote by $P_j(\omega') = \langle \int_z \omega' \mid z \in i_* H_1(\Delta_j) \rangle$ the group of periods of ω' on the set Δ_j. Consider the set

$$\mathcal{R}_\Delta \equiv \{\mathrm{rk}_{\mathbb{Q}} \langle P_1(\omega'), \dots, P_k(\omega') \rangle \mid \omega' \in [\omega]\}.$$

By Proposition 3.5, for $\omega' \in [\omega]$ it holds that $\omega' = f(x)\omega$ for some admissible function $f(x)$. Since by Proposition 3.10 it holds that $f|_{\Delta_j} = c_j \in \mathbb{R} \setminus 0$, we obtain $P_j(\omega') = c_j P_j(\omega)$. Then

$$\mathcal{R}_\Delta = \{\mathrm{rk}_{\mathbb{Q}} \langle c_1 P_1, \dots, c_k P_k \rangle \mid c_j \in \mathbb{R} \setminus 0\}, \tag{10}$$

where $P_j = P_j(\omega)$.

The study of the structure of the set (10) is a number theory problem. Here we shall only touch upon some its properties.

The set $\mathcal{R}_\Delta$ is bounded: $\mathcal{R}_\Delta \subseteq [r_\Delta, R_\Delta]$, where $r_\Delta = \min(\mathcal{R}_\Delta)$ and $R_\Delta = \max(\mathcal{R}_\Delta)$.[2] Define $r_j = \mathrm{rk}_{\mathbb{Q}} P_j$.

Proposition 4.2 *For non-compactifiable foliations the following hold:*

(i) *Lower bound:* $r_\Delta \geq \max_j r_j \geq 2$,
(ii) *Upper bound:* $R_\Delta = \sum_j r_j \geq 2k$,
(iii) *Range:* $k - 1 \leq R_\Delta - r_\Delta \leq b_1(M) - 2 - c(\omega)$,

where k is defined by (9)*; for generic Morse forms, $k = m(\omega)$.*

Obviously, for a compactifiable foliation, $\mathcal{R}_\Delta = \{0\}$.

PROOF. That $k = m(\omega)$ has been shown as Lemma 4.1.

(i) Since $\mathrm{rk}_{\mathbb{Q}}(c_j P_j) = r_j$, by (10) we have $r_\Delta \geq \max_j r_j$. A minimal component contains at least two (homologically independent in M) 1-cycles with incommensurable periods [13], i.e., $r_j \geq 2$.

(ii) For each Δ_j, let periods $\{\alpha_1^{(j)}, \ldots, \alpha_{r_j}^{(j)}\}$ be independent over $\mathbb{Q}$; then so are

$$\{c_1\alpha_1^{(1)}, \ldots, c_1\alpha_{r_1}^{(1)},\ c_2\alpha_1^{(2)}, \ldots, c_2\alpha_{r_2}^{(2)},\ \ldots,\ c_k\alpha_1^{(k)}, \ldots, c_k\alpha_{r_k}^{(k)}\}$$

for suitable $c_l \in \mathbb{R}$. Thus $R_\Delta = \sum_{j=1}^{k} r_j$.

(iii) Choosing $c_j = 1/\alpha_1^{(j)}$, we have $r_\Delta \leq \sum_{j=1}^{k} \mathrm{rk}(c_j P_j) - k + 1$, which by (ii) gives $R_\Delta - r_\Delta \geq k - 1$. For the upper bound in (iii), see Theorem 4.11. □

The following example shows that strict inequality can hold in (i):

Example 4.3 *It is possible that $r_\Delta > \max_j r_j$. Indeed, represent the double torus M_2^2 as a connected sum of two tori T^2 connected by a tube; see Fig. 4. Let the foliation be defined by the form $dx + \sqrt{2}\,dy$ on the first torus, $dx + \sqrt{3}\,dy$ on the second, and be compact on the tube.*

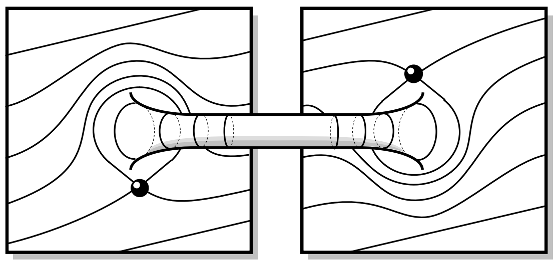

Fig. 4. Two tori T^2 with a hole, each one with its own irrational winding and compact leaves around the hole. They are glued together by the holes into a double torus M_2^2 with two minimal components and a maximal component on the tube.

[2] By $\min(A)$ we denote $\min_{x \in A} x$, and similarly for $\max(A)$.

Then Δ has two connected components, with their groups of periods $P_1 = \langle 1, \sqrt{2}\rangle$ and $P_2 = \langle 1, \sqrt{3}\rangle$. We have $\mathrm{rk}\langle P_1, P_2\rangle = 3$, $\mathrm{rk}\langle P_1, \sqrt{5}P_2\rangle = 4$; $\mathcal{R}_\Delta = \{3, 4\}$ *and* $r_\Delta = 3$, *while* $\mathrm{rk}\, P_j = 2$.

In a general position case (i.e., if all periods are incommensurable) the range is the smallest possible: $R_\Delta - r_\Delta = k - 1$, and all intermediate values are reached: $\mathcal{R}_\Delta = [r_\Delta, R_\Delta] \subset \mathbb{Z}$. In some special cases, however, the set $\mathcal{R}_\Delta$ can have gaps:

Example 4.4 *Let M_2^2 be as in the previous example, but the forms on both tori be $dx + \sqrt{2}\,dy$. Then $P_1 = P_2 = \langle 1, \sqrt{2}\rangle$;* $\mathrm{rk}\langle P_1, P_2\rangle = 2$, $\mathrm{rk}\langle P_1, \sqrt{5}P_2\rangle = 4$; $\mathcal{R}_\Delta = \{2, 4\}$; *cf. Proposition 4.5.*

Existence of gaps is connected with the algebraic structure of the groups of periods. In the previous example the space generated by the periods $\langle 1, \sqrt{2}\rangle_{\mathbb{Q}}$ was a field; this can be generalized:

Proposition 4.5 *Let $\Delta = \Delta_1 \cup \Delta_2$ and $P_1 = P_2 = P$, where P is a field. Then $\mathcal{R}_\Delta = \{\mathrm{rk}\, P,\ 2\,\mathrm{rk}\, P\}$ has a gap.*

PROOF. Consider $c \in \mathbb{R}$. If $P \cap cP = 0$ then $\mathrm{rk}\langle P, cP\rangle = 2\,\mathrm{rk}\, P$. Otherwise there exists $x \in P$ such that $x = cy \neq 0$ for some $y \in P$; therefore $c = xy^{-1} \in P$, $cP = P$, and $\mathrm{rk}\langle P, cP\rangle = \mathrm{rk}\, P$. We have $\mathcal{R}_\Delta = \{\mathrm{rk}\, P,\ 2\,\mathrm{rk}\, P\}$. Since $\mathrm{rk}\, P \geq 2$ (cf. Proposition 4.2 (i)), $\mathcal{R}_\Delta$ has a gap. □

The condition for P to be a field is important. Indeed, change the form in Example 4.4 to $dx + \sqrt[3]{2}\, dy$. Then $P = \langle 1, \sqrt[3]{2}\rangle_{\mathbb{Q}}$ is not a field. Taking $c \in \{1,\ 1/\sqrt[3]{2},\ \sqrt{3}\}$, we get $\mathcal{R}_\Delta = \{2, 3, 4\}$, which has no gaps.

4.2 The set $\mathcal{R}_H$

Consider now the part DH_ω of (8) and calculate

$$\mathcal{R}_H \equiv \{\mathrm{rk}_{\mathbb{Q}}\, \langle \textstyle\int_z \omega' \mid z \in DH_\omega\rangle \mid \omega' \in [\omega]\}.$$

Proposition 4.8 below shows that this set is well-defined and does not depend on the choice of the duality map D.

While in minimal components the form almost cannot be changed without changing the foliation, maximal components allow one to vary the corresponding periods arbitrarily:

Lemma 4.6 *Let ω be a Morse form, $\mathcal{C} = \mathcal{C}^{max} \subseteq M$ a maximal component of $\mathcal{F}_\omega$, and $\alpha : [0,1] \to \overline{\mathcal{C}}$, $\alpha(0), \alpha(1) \in \partial\mathcal{C}$, a curve transverse to leaves. Then for any $A \in \mathbb{R}$ there exists a (not necessarily Morse) form ω' such that $\overline{\mathcal{F}}_{\omega'}|_{\mathcal{C}} = \mathcal{F}_\omega|_{\mathcal{C}}$, $\omega'|_{M\setminus\mathcal{C}} = \omega$, and $\int_\alpha \omega' = A$. If $\operatorname{sgn} \int_\alpha \omega = \operatorname{sgn} A$, then ω' can be chosen a Morse form.*

The condition $\overline{\mathcal{F}}_{\omega'} = \mathcal{F}$ means that ω' is collinear with ω and $\operatorname{int}(\operatorname{Sing}\omega') = \emptyset$. If ω' is also a Morse form, then $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$. Otherwise, however, $\operatorname{Sing}\omega'$ as constructed below can include two whole leaves of ω.

PROOF. Choose $J = [t_1, t_2] \subset (0, 1)$. Consider a smooth function $g(t)$ such that $g|_{\mathbb{R}\setminus J} \equiv 1$, $g(t) = 0$ at two points, if any, and $g|_J$ grows large enough, or drops low enough (negative if needed), to make $\int_0^1 g(t)\,\omega(d\alpha(t)) = A$. It induces on M a function $f(x)$ constant on leaves, such that $f|_{M\setminus\mathcal{C}} \equiv 1$ and $f(\alpha(t)) = g(t)$ in $\mathcal{C}$. Then $\omega' = f(x)\omega$ has the desired properties.

Let now $\operatorname{sgn} \int_\alpha \omega = \operatorname{sgn} A$; assume $A > 0$. Choose t_1, t_2 above such that $\left(\int_0^{t_1} + \int_{t_2}^1\right) \omega(d\alpha(t)) < A$. Then $g(t)$ can be chosen positive. By Proposition 3.10, $f(x)$ is admissible; thus ω' is a Morse form. □

Corollary 4.7 *Let $\mathcal{C}_i = \mathcal{C}_i^{max}$ be maximal components of $\mathcal{F}_\omega$, $\alpha_i : [0,1] \to \overline{\mathcal{C}_i}$, $\alpha_i(0), \alpha_i(1) \in \partial\mathcal{C}_i$, be curves transverse to leaves, and $A_i \subset \mathbb{R} \setminus 0$, $\operatorname{sgn} A_i = \operatorname{sgn} \int_{\alpha_i} \omega$. Then there exists a Morse form $\omega' \in [\omega]$ such that $\int_{\alpha_i} \omega' = A_i$ and $\omega' \equiv \omega$ on $M \setminus (\bigcup \mathcal{C}_i)$.*

That is, by choosing a suitable form in $[\omega]$, the absolute values of the integrals along the edges of the foliation graph $\overrightarrow{\Gamma}_\omega$ can be varied in any desired way.

Proposition 4.8 *It holds that*[3]

$$\mathcal{R}_H = [a, c(\omega)], \tag{11}$$

where

$$a = \begin{cases} 0 & \text{if } \overrightarrow{\Gamma}_\omega \text{ is acyclic,} \\ 1 & \text{otherwise} \end{cases} \tag{12}$$

and $c(\omega)$ is the number of homologically independent compact leaves.

Note that we only require $\overrightarrow{\Gamma}_\omega$ to have no (directed) circuits, while it may have semi-circuits, i.e., the undirected graph Γ_ω does not have to be acyclic.

[3] When appropriate, we use the notation $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$.

PROOF. The upper bound follows from (4) since $c(\omega) = \operatorname{rk} H_\omega$. The lower bound follows from the fact that if the foliation graph $\overrightarrow{\Gamma}_\omega$ has a circuit containing a closed curve α, then $\langle \int_\alpha \omega' \rangle \neq 0$ and thus $\mathcal{R}_H \not\ni 0$. It remains to show that the bounds are exact and all intermediate values are reached.

Recall from Lemma 2.3 that $H_\omega = \langle [\gamma_1], \ldots, [\gamma_c] \rangle$, $c = c(\omega)$, and $D[\gamma_i]$ form a basis of DH_ω with $D[\gamma_i] \cdot [\gamma_j] = \delta_{ij}$. Realize the cycles $D[\gamma_i]$ by closed curves α_i. Recall that $\mathcal{C}_j^{max} \cong \gamma_j \times (0,1)$; without loss of generality we can assume that $\alpha_i \cap \mathcal{C}_j^{max} \neq \emptyset$ iff $i = j$, all α_i are transverse to leaves, and each $\alpha_i \cap \mathcal{C}_i^{max}$ is connected [6, 14].

By Corollary 4.7, we can slightly vary the integrals of ω' along α_i, $i = 1, \ldots, c$, and therefore vary $r(\omega') = \operatorname{rk}_{\mathbb{Q}}\{\int_{\alpha_1} \omega', \ldots, \int_{\alpha_c} \omega'\}$ between 1 and c. If the foliation digraph $\overrightarrow{\Gamma}_\omega$ is acyclic, by Lemma 2.4 we can construct an admissible function $f(x)$ such that $\int_{\alpha_i} f(x)\omega = 0$ for all i, which adds 0 to $\mathcal{R}_H$. Otherwise $\langle \int_{\alpha_i} \omega' \rangle \neq 0$, i.e., $0 \notin \mathcal{R}_H$. □

Remark 4.9 *Sometimes the undirected graph Γ_ω can give information about $a = \min(\mathcal{R}_H)$. Obviously, if Γ_ω is acyclic, then $a = 0$. If Γ_ω has loops, has fewer than two vertices of degree 1, or has a vertex incident via multiple edges to only one vertex, then $a = 1$. More generally, if Γ_ω has a subgraph without vertices of degree 1, which is connected to the rest of Γ_ω by a cut edge or a cut vertex, then $a = 1$.*

4.3 The set $\mathcal{R}$

In their influential paper [5], Arnoux and Levitt introduced the *first non-commutative Betti number* $b_1'(M)$—the maximum rank of a (non-Abelian) free quotient group of the fundamental group $\pi_1(M)$ [13].

Recall that $c(\omega)$ is the number of homologically independent compact leaves and $m(\omega)$ is the number of minimal components. We have previously shown [15] that

$$c(\omega) + m(\omega) \leq b_1'(M), \tag{13}$$

and on any manifold there exists a form for which equality holds.

Unfortunately, we are not aware of any practical methods of calculating $b_1'(M)$ for a specific manifold. We can, however, bound $b_1'(M)$ from above by a value $h(M) \leq b_1(M)$, which can be nicely calculated for many specific manifolds; see formulas in [7] also reproduced in [6].

Proposition 4.10 *It holds that*

$$b_1'(M) \leq h(M),$$

where $h(M)$ is the maximum rank of a subgroup in $H^1(M,\mathbb{Z})$ with trivial cup-product $\smile : H^1(M,\mathbb{Z}) \times H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})$.

This follows from exactness of the bound (13), while $c(\omega)+m(\omega) \le h(M)$ [14].

The proposition shows that in practice one can consider $h(M)$ in upper bounds involving $b_1'(M)$, such as Theorem 4.11 or Corollary 5.1 (ii) below.

We can now summarize our results as follows:

Theorem 4.11 *Let $\mathcal{R} = \{\operatorname{rk}\omega' \mid \omega' \in [\omega]\}$. Then*[4]

$$\mathcal{R} = \begin{cases} [a, c(\omega)], & \text{if } \mathcal{F}_\omega \text{ is compactifiable,} \\ \mathcal{R}_\Delta + [0, c(\omega)], & \text{otherwise;} \end{cases} \tag{14}$$

$$\max(\mathcal{R}) - \min(\mathcal{R}) \le \begin{cases} b_1'(M) - a, & \text{if } \mathcal{F}_\omega \text{ is compactifiable,} \\ b_1(M) - 2, & \text{otherwise,} \end{cases} \tag{15}$$

where a is given by (12) *and $\mathcal{R}_\Delta$ is given by* (10)*. In particular,*

$$\min(\mathcal{R}) = \begin{cases} a, & \text{if } \mathcal{F}_\omega \text{ is compactifiable,} \\ r_\Delta \ge 2, & \text{otherwise;} \end{cases} \tag{16}$$

$$\max(\mathcal{R}) = \begin{cases} c(\omega), & \text{if } \mathcal{F}_\omega \text{ is compactifiable,} \\ R_\Delta + c(\omega), & \text{otherwise,} \end{cases} \tag{17}$$

where r_Δ and R_Δ are described by Proposition 4.2.

Note that in the non-compactifiable case, not all intermediate values between (16) and (17) are guaranteed to be reached; cf. Proposition 4.5.

PROOF. If $\mathcal{F}_\omega$ is compactifiable, then $\mathcal{R}_\Delta = \emptyset$. In this case (14) is given by (11) and then (15) follows from (13) given that $m(\omega) = 0$. Assume now that $\mathcal{F}_\omega$ has minimal components.

Then $\mathcal{R}_\Delta \ne \emptyset$ by Proposition 4.2 (i). If $c(\omega) = 0$, then by (8) we have $\mathcal{R} = \mathcal{R}_\Delta$; so assume $c(\omega) \ne 0$. We can vary the form in each maximal component and each set Δ_j independently. Fixing ω' in Δ, by Lemma 4.6 we can choose ω' in maximal components such that $\int_z \omega' \in \langle P_j(\omega') \rangle$, $z \in DH_\omega$. Together with Proposition 4.8 this gives (14); then (17) and the equality in (16) are obvious.

[4] We define $A + B = \{a + b \mid a \in A,\, b \in B\}$.

Finally, by definition $\max(\mathcal{R}) \le b_1(M)$. Proposition 4.2 (i) gives the lower bound in (16) and then the upper bound in (15). $\square$

For compactifiable foliations, this theorem generalizes Theorem 4.1 from [14].

5 Corollaries in terms of manifold and foliation structure

Since $\mathcal{F}_\omega$ is an invariant of the collinearity class $[\omega]$, Theorem 4.11 allows one to connect the foliation topology with $\mathcal{R}$.

Corollary 5.1 *In terms of the* manifold *structure:*

(i) $\min(\mathcal{R}) \le 1$ *iff the foliation is compactifiable. If* $1 \in \mathcal{R}$*, then the foliation has homologically non-trivial compact leaves.*
(ii) $\max(\mathcal{R}) > b_1'(M)$ *implies that the foliation is non-compactifiable, i.e., has a minimal component.*

The latter follows from (15) and (16).

Corollary 5.2 *In terms of the* foliation *structure,* $|\mathcal{R}| = 1$ *in the following cases:*

(i) $\mathcal{R} = \{0\}$ *iff* $\mathcal{F}_\omega$ *is compactifiable and all its compact leaves are homologically trivial:* $c(\omega) = 0$*. In particular, if* $b_1(M) = 0$*, then* $\mathcal{R} = \{0\}$*.*
(ii) $\mathcal{R} = \{1\}$ *iff* $\mathcal{F}_\omega$ *is compactifiable,* $c(\omega) = 1$*, and* $\overrightarrow{\Gamma}_\omega$ *has a (unique) circuit.*
(iii) $\mathcal{R} = \{r\}$*,* $r \ge 2$*, iff* $\mathcal{F}_\omega$ *has minimal components, all of them connected by singular leaves:* $k = 1$ *in* (9)*, and all its compact leaves are homologically trivial:* $c(\omega) = 0$*. In particular, these conditions hold for minimal foliations.*

Note that in (iii), $\mathcal{F}_\omega$ can contain several minimal components in spite of $k = 1$.

PROOF. Cases (i) and (ii) follow from (11), (12), and Corollary 5.1 (i).

(iii) Let $c(\omega) = 0$ and $k = 1$. By (16) and (17) we have $\mathcal{R} \subseteq [r_\Delta, R_\Delta]$. By Proposition 4.2 (ii), (i), $R_\Delta = r_\Delta \ge 2$.

Conversely, let $\mathcal{R} = \{r\}$, $r \ge 2$. By Corollary 5.1 (i) we have $m(\omega) \ge 1$ (thus $k \ge 1$), and by (16), (17) it holds $r = r_\Delta = R_\Delta + c(\omega)$. Since $r_\Delta \le R_\Delta$, this implies $c(\omega) = 0$ and $r_\Delta = R_\Delta$. Then Proposition 4.2 (iii) gives $k \le 1$. $\square$

The topology of foliations that can be defined by a form of maximal possible rank for a given M, $\operatorname{rk}\omega = b_1(M)$, is tightly connected with the structure of the cup-product:

Theorem 5.3 *Assume* $b_1(M) \in \mathcal{R}$*. Then* $\operatorname{rk}\ker\smile\ \geq c(\omega)$*.*

PROOF. Denote by $[\gamma]$ the homology class of a compact leaf, $[\varphi]$ the cohomology class of a form φ, and D a Poincaré duality.

Let $\operatorname{rk}\omega = b_1(M)$ and $\gamma \in \mathcal{F}_\omega$, $[\gamma] \neq 0$. We will construct a (non-Morse) form φ such that $[\varphi] = D[\gamma]$ and $n[\varphi] \in \ker\smile$ for some $n \in \mathbb{Z} \setminus 0$.

Namely, consider a (non-Morse) form φ such that $\mathcal{F}_\varphi = \mathcal{F}_\omega$ in a cylindrical neighborhood of γ and $\varphi \equiv 0$ outside it; thus $\varphi \wedge \omega = 0$. The form φ can be chosen such that its cohomology class $[\varphi] \in H^1(M, \mathbb{Z})$ and $[\varphi] \neq 0$. Thus $[\varphi] \smile_{\mathbb{R}} [\omega] = \sum \alpha_i([\varphi] \smile \xi_i) = 0$, where $\smile_{\mathbb{R}}$ is the cup-product on $H^1(M, \mathbb{R})$ and $\{\xi_i\}$ is a basis in $H^1(M, \mathbb{Z})$. Define $u_i = [\varphi] \smile \xi_i$.

Since $\operatorname{rk}\omega = b_1(M)$, all α_i are independent over $\mathbb{Q}$, so $\sum \alpha_i u_i = 0$ implies that all u_i belong to the torsion of $H^2(M, \mathbb{Z})$. Therefore for some $0 \neq n \in \mathbb{Z}$ we have $n[\varphi] \smile \xi_i = 0$ for all i; thus $n[\varphi] \in \ker\smile$. Since $H^1(M, \mathbb{Z})$ has no torsion, $n[\varphi] \neq 0$.

Now, consider compact leaves $\gamma_1, \ldots, \gamma_{c(\omega)}$ from (1) and the corresponding φ_i as above such that $n_i[\varphi_i] \in \ker\smile$ for some $n_i \in \mathbb{Z} \setminus 0$. Since the $[\gamma_1], \ldots, [\gamma_{c(\omega)}]$ are independent, so are $n_i[\varphi_i]$; thus $\operatorname{rk}\ker\smile\ \geq c(\omega)$. □

Corollary 5.4 *Assume* $b_1(M) \in \mathcal{R}$ *and* $\mathcal{F}_\omega$ *is compactifiable. Then* $c(\omega) = b_1(M)$ *and* $\smile\ \equiv 0$*.*

Indeed, by (17), $c(\omega) = b_1(M)$; Theorem 5.3 gives $\smile\ \equiv 0$.

Corollary 5.5 *Assume* $b_1(M) \in \mathcal{R}$*,* $b_1(M) \neq 0$*. If* $\smile$ *is non-degenerate, then* $\mathcal{F}_\omega$ *has a minimal component:* $m(\omega) \geq 1$*, and all its compact leaves are homologically trivial:* $c(\omega) = 0$*.*

This follows from Theorem 5.3 and Corollary 5.4.

Recall from Proposition 4.2 (ii) that $\sum \operatorname{rk}\omega|_{\Delta_j} \leq \max(\mathcal{R}) \leq b_1(M)$, where Δ_j are those connected components of the union Δ of all non-compact leaves and singularities that contain minimal components. The condition $b_1(M) \in \mathcal{R}$ from the last several statements does not necessarily require the rank of a given form ω to be large; e.g., $\sum \operatorname{rk}\omega|_{\Delta_j} = b_1(M)$ implies $b_1(M) \in \mathcal{R}$ even if $\operatorname{rk}\omega$ is small. In particular:

Corollary 5.6 *If* $\sum \operatorname{rk}\omega|_{\Delta_j} = b_1(M)$*, then all compact leaves of* $\mathcal{F}_\omega$ *are homologically trivial:* $c(\omega) = 0$.

This follows from (17) and Proposition 4.2 (ii).

Example 5.7 *Consider* $M_g^2 = \sharp_{j=1}^g T_j^2$*, a connected sum of* g *tori, and a form* ω *that has a minimal component in each torus, but with the same group of periods* $\langle 1, \sqrt{2}\rangle$*; cf. Fig. 4. Then* $\sum \operatorname{rk}\omega|_{\Delta_j} = 2g$ *and thus* $c(\omega) = 0$*, even though* $\operatorname{rk}\omega = 2$.

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