Realization of a graph as the Reeb graph of a Morse, Morse–Bott or round function

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Contents

Introduction

Concept of Reeb graph

- **Reeb graph** was introduced by George Reeb (1946), in that time
  - only for simple Morse functions
  - on closed manifolds

  as a **quotient space**: connected components of level sets contracted to points
- He noted that it is a 1-dim (CW) complex: a **finite graph** (with multiple edges)
  - for the type of functions he considered
  - this is used in all modern applications of the Reeb graph
- Alexander Kronrod (1950) introduced **tree** of connected components of level sets
  - for continuous functions
  - on a sphere

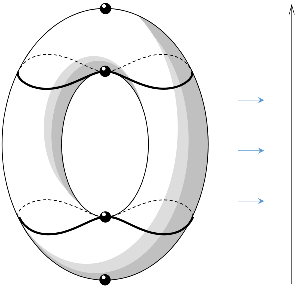
  This tree can be infinite.
- Reeb graph also is called the **Kronrod–Reeb graph**.

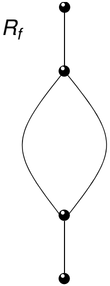
Reeb graph of a smooth function

- X is a topological space; $f : X \to \mathbb{R}$ is a continuous function
- **Contour** of f: connected component of its level set $f^{-1}(y)$
- $x \sim y$ is an equivalence relation: $x, y \in$ the same contour of f

Definition

The **Reeb graph** R_f is the quotient space $X/\sim$, endowed with the quotient topology.
For smooth functions: image of a critical contour is called a **vertex**.





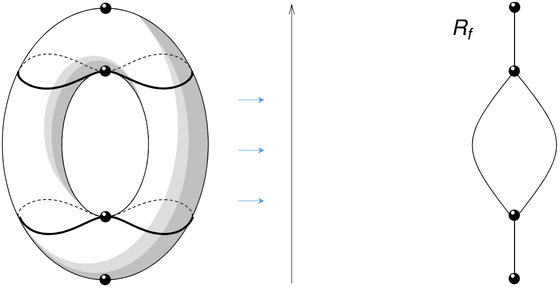
Geometric meaning of the Reeb graph

This graph shows the **evolution** of the level sets:

- contours can **split** into two or more
- contours can **merge** into one

In some “good” cases, Reeb graph is indeed a **graph**

- non-vertices form **edges**
- much more on this, later

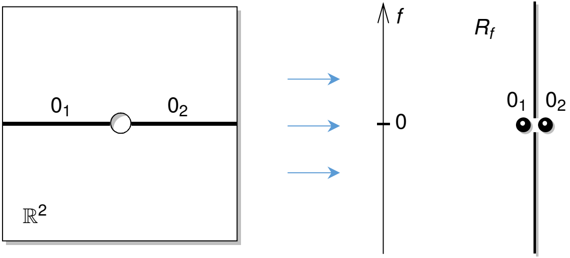


Counterexample: Reeb graph is not a finite graph

Generally, the Reeb graph is not a graph.
This quotient space can be ill-behaved even for very good functions:

Example

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and $f(x,y) = y$ be the projection.
Then R_f is the line with two origins (bug-eyed line).
Not a graph, even non-Hausdorff.



The problem is that the manifold is **not compact**.

Reeb graph as a finite graph

- M is a **manifold**, $f: M \to \mathbb{R}$ is a **smooth function**
- A finite **graph** can have multiple edges and loops: a 1-dimensional CW complex

Definition

The Reeb graph R_f **has the structure of a finite graph** G, if there is a homeomorphism $h: R_f \to G$ mapping vertices of R_f bijectively to vertices of G.

We will say that R_f is **isomorphic** to G or just R_f **is** G (abuse of language).

Example

- The Reeb graph of a simple Morse function has the structure of a finite graph [Reeb (1946)]
- The Reeb graph of a function f with finite $\mathrm{Crit}(f)$ has the structure of a finite graph [Sharko (2006)]
- The Reeb graph of a simple Morse–Bott function on a surface has the structure of a finite graph [Martínez-Alfaro et al. (2016)]

When is the Reeb graph a finite graph?

Theorem (Saeki (2021))

Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function. Then:
R_f has the structure of a finite graph $\Leftrightarrow$ f has a finite number of critical values.

This makes it possible:

- to work with a wide class of functions, including Morse–Bott and round functions;
- to study these functions using graph theory.

Realization problem:
When is a finite graph the Reeb graph of a function?

Realization: smooth function

Realization problem: Is any finite graph the Reeb graph of some function?

No. But yes for graphs **without loops** (edge with both endpoints at the same vertex):

Theorem (Masumoto and Saeki (2011))

Let G be a finite graph. Then:
there is a smooth function $f : M \to \mathbb{R}$ such that R_f is $G \Leftrightarrow G$ has no loops.

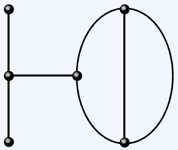
Indeed, R_f that is a finite graph has an acyclic orientation $\Rightarrow$ no loops.

Realization: Morse function. Counterexample

Realization problem in some **class of functions**: additional conditions on the graph.

Example (Sharko (2006))

Not R_f of any Morse function. Not even function with finite $\mathrm{Crit}(f)$:



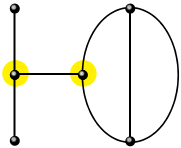
It is not R_f of a Morse–Bott or round function.

Why? Let's see. First, some graph theory...

Some graph theory

Definition

- **Cut vertex**: $G \setminus v$ has more connected components. Isolated vertex is not.



2 cut vertices, 4 blocks, 3 of them leaf blocks

Some graph theory

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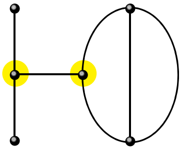
- **Cut vertex**: $G \setminus v$ has more connected components. Isolated vertex is not.
- **Biconnected** graph: connected, without cut vertices.



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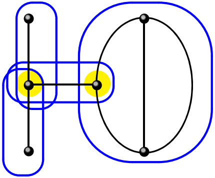

2 cut vertices, **4 blocks**, 3 of them leaf blocks

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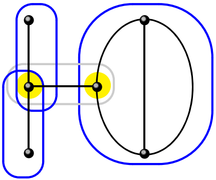
- **Biconnected** graph: connected, without cut vertices.
- **Block** of a graph: maximal biconnected subgraph. Isolated vertex is a block.
- **Leaf block**: a block with at most one cut vertex.

Blocks are attached to each other at shared vertices = cut vertices of the graph. (This forms the **block-cut tree**, of which leaf blocks are leafs—hence the term.)



2 cut vertices, 4 blocks, **3 of them leaf blocks**

Realization: Morse function

- Closed manifold
- Morse function
- Generally, function with finite number of critical points

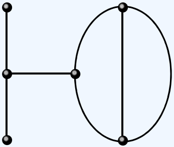
Theorem (Michalak (2018) + Gelbukh (submitted1))

G is R_f of a smooth function with finite $\mathrm{Crit}(f)$ on a closed manifold $\Leftrightarrow$
G is finite, no loops, all leaf blocks are •——• (P_2).

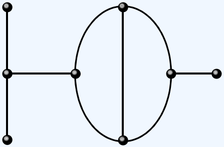
f can be chosen Morse.

To make a given graph realizable by a Morse f, add P_2 to each non-P_2 leaf block:

Example



☹ non-P_2 leaf block



☺ all P_2 leaf blocks

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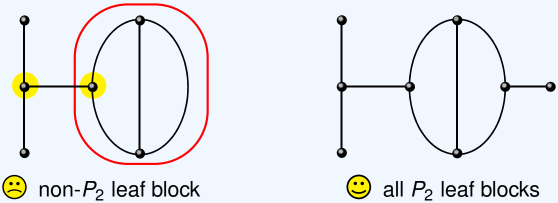

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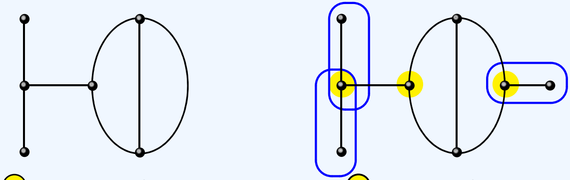

☹ non-P_2 leaf block ☺ all P_2 leaf blocks

Realization: Morse–Bott function

Morse–Bott function; generally, function with finite number of critical **submanifolds**:

Theorem (Gelbukh (submitted2))

For any given $n \geq 2$*,*
G *is* R_f *of a smooth* f *with* $\mathrm{Crit}(f) =$ *finite no. of submanifolds, on closed n-manifold* $\Leftrightarrow$
G *is finite, no loops, and*

- *each leaf block* L *has a vertex* v *with* $\deg_G(v) \leq 2$*,*
- *two such vertices if* L *is a non-trivial (has an edge) connected component of* G*.*

f *can be chosen Morse–Bott.*

To make G realizable by a Morse–Bott f, subdivide an edge in leaf blocks where missing:

Example

☹ no vertex of degree ≤ 2 ☺ all leaf blocks have ≤ 2

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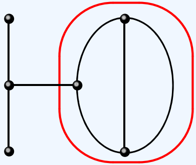
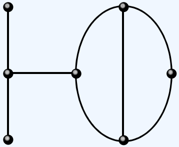


no vertex of degree ≤ 2



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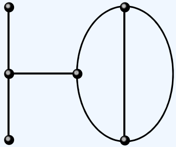

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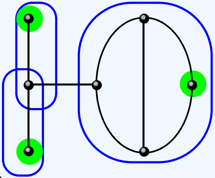


☺ all leaf blocks have ≤ 2

Realization: Morse–Bott function (homeomorphism)

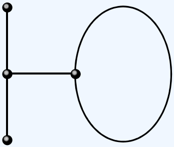
Morse–Bott functions play a special role in the Reeb graph theory:

Theorem (Gelbukh (in press))

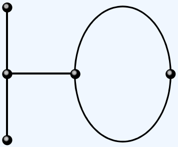
Any *finite graph is* ***homeomorphic*** *to the Reeb graph of a Morse–Bott function.*

True even for a graph with loops: can subdivide a loop by a vertex of degree 2.

Example



☹ loop: no any function



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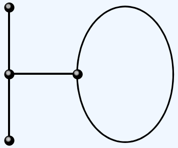

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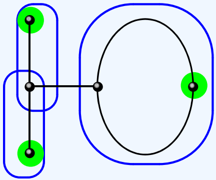


☺ no loop, Morse–Bott function

Realization: round function

Definition

Round function: smooth function $f : M \to \mathbb{R}$ on a closed manifold M, with $\mathrm{Crit}(f) = \bigcup S^1$, a finite number of disjoint **circles**.

This time, the structure of R_f depends on manifold:

- dimension
- whether orientable

Theorem (Gelbukh (submitted2))

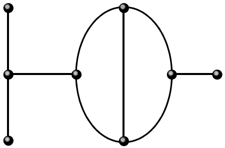
G is R_f of a round function on $M^n \Leftrightarrow G$ is finite, no loops, and

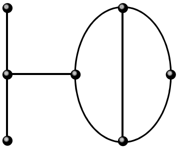
$$\textit{each leaf block} \begin{cases} \textit{has a non-cut vertex of } \deg v = 2 & \textit{if } n = 2\textit{, orientable surface} \\ \textit{has a non-cut vertex of } \deg v \leq 2 & \textit{if } n = 2\textit{, non-orientable surface} \\ \textit{is } \bullet\!\!-\!\!-\!\!-\!\!\bullet \ (P_2) & \textit{if } n \geq 3 \end{cases}$$

Realization: conclusion

A **graph** can be realized by functions of **different classes** and on **different manifolds**:

$L_i(G)$ leaf blocks, | Morse, Morse–Bott, | $\dim M^n$, orientability,
$b_1(G)$ cycle rank | round | $corank(\pi_1(M^n))$





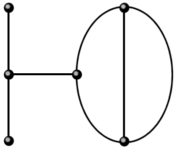
f	$n=2$ orient	$n=2$ non-or	$n \geq 3$
Morse	+	+	+
Morse–Bott	·	·	·
round	·	+	+

f	$n=2$ orient	$n=2$ non-or	$n \geq 3$
Morse	·	·	·
Morse–Bott	+	+	+
round	·	+	·

Since $\mathrm{corank}(\pi_1(M)) \geq b_1(G)$ (Gelbukh (2019)): surface genus $\geq \begin{cases} 2 & \text{orientable,} \\ 4 & \text{non-orientable.} \end{cases}$

Realization of the Sharko graph

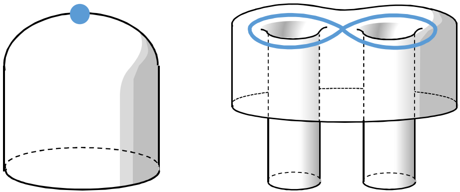
What functions realize the Sharko graph?



Example

On an orientable surface, these functions have two types of extrema:

- isolated points,
- wedge sum $S^1 \vee S^1$.



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Thank you! :)

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