

**THE NUMBER OF MINIMAL COMPONENTS AND  
HOMOLOGICALLY INDEPENDENT COMPACT LEAVES OF  
A WEAKLY GENERIC MORSE FORM ON A CLOSED SURFACE**

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ABSTRACT. On a closed orientable surface  $M_g^2$  of genus  $g$ , we consider the foliation of a weakly generic Morse form  $\omega$  on  $M_g^2$  and show that for such forms  $c(\omega) + m(\omega) = g - \frac{1}{2}k(\omega)$ , where  $c(\omega)$  is the number of homologically independent compact leaves of the foliation,  $m(\omega)$  is the number of its minimal components, and  $k(\omega)$  is the total number of singularities of  $\omega$  that are surrounded by a minimal component. We also give lower bounds on  $m(\omega)$  in terms of  $k(\omega)$  and the form rank  $\text{rk } \omega$  or the structure of  $\ker[\omega]$ , where  $[\omega]$  is the integration map.

1. INTRODUCTION

Consider a closed connected orientable smooth two-dimensional manifold  $M = M_g^2$  of genus  $g$ . Let  $\omega$  be a Morse form on  $M$ , i.e., a closed 1-form with Morse singularities  $\text{Sing } \omega$ , locally the differential of a Morse function. This form defines a foliation  $\mathcal{F}_\omega$  on  $M \setminus \text{Sing } \omega$ . A leaf  $\gamma \in \mathcal{F}_\omega$  is called compactifiable if  $\gamma \cup \text{Sing } \omega$  is compact.

A Morse form is called *generic* if each of its non-compact compactifiable leaves is compactified by a unique singularity [2, Definition 9.1]. The set of such forms is dense in any cohomology class [2, Lemma 9.2]. The term *generic* introduced in [2] is somewhat misleading because the set of such forms is not open. We find it plausible that such forms are the “majority” of Morse forms and thus their properties are in a sense “typical,” though we are not aware of any proof of this.

Our results hold for a wider class of forms, which we call *weakly generic*: the requirement for a leaf to be compactified by only one singularity is only applied to the leaves not surrounded by minimal components.

The number  $m(\omega)$  of minimal components and  $c(\omega)$  of homologically independent compact leaves are important topological characteristics of the foliation. On  $M_g^2$  it holds [5]

$$(1) \quad 0 \leq c(\omega) + m(\omega) \leq g$$

and all such combinations are possible on a given  $M$  [4]. In particular, if  $c(\omega) = g$  then the foliation is compactifiable, i.e.,  $m(\omega) = 0$ , though the converse is not true: there exist compactifiable foliations with  $c(\omega) < g$ .

In this paper, for weakly generic forms we give a precise expression for  $c(\omega) + m(\omega)$  and better bounds on  $m(\omega)$ . A useful characteristic of a weakly generic form foliation is the number  $k(\omega)$  of singularities that are surrounded by a minimal

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component; for a weakly generic form  $k(\omega)$  is even (Corollary 7). Our main result states that for such forms the inequality (1) becomes

$$(2) \quad c(\omega) + m(\omega) = g - \frac{k(\omega)}{2}$$

(Theorem 5). In particular, for weakly generic forms on  $M_g^2$ ,  $g \neq 0$ , the exact lower bound in (1) is

$$1 \leq c(\omega) + m(\omega) \leq g$$

(Corollary 6). On the other hand, (2) gives a criterion for compactifiability for weakly generic forms [11]:  $m(\omega) = 0$  iff  $c(\omega) = g$ .

The inequality (1) gives an upper bound on the number of minimal components:  $m(\omega) \leq g$ ; this was also proved in [9]. For weakly generic forms, (2) gives a better upper bound:

$$(3) \quad m(\omega) \leq g - \frac{k(\omega)}{2}.$$

We are not aware, though, of any *lower* bound on  $m(\omega)$  given in literature, except for that if  $\text{rk } \omega > g$  (the rank of the group of periods) then the foliation has minimal components:  $m(\omega) > 0$  [11]. For weakly generic forms, we give a lower bound on  $m(\omega)$ , cf. (3):

$$(4) \quad m(\omega) \geq g - \frac{k(\omega)}{2} - h(\ker[\omega])$$

(Theorem 10). Here,  $\ker[\omega] = \langle z \in H_1(M) \mid \int_z \omega = 0 \rangle$  and  $h(*)$  is the rank of a maximal subgroup consisting of non-intersecting cycles. We calculate the value of  $h(\ker[\omega])$  (Lemma 8) and bound it in terms of  $\text{rk } \ker[\omega]$  (Corollary 9).

The bound (4) is not exact; however, it becomes exact together with a trivial observation that  $m(\omega) > 0$  if  $k(\omega) > 0$ . All intermediate values are also reached, except for  $m = 1$  when  $k = 0$  and  $h(\ker[\omega]) = g$ ; this combination is impossible [6]. Our account of the relationships between  $g$ ,  $k(\omega)$ ,  $h(\ker[\omega])$ , and  $m(\omega)$  is complete: we build a (generic) form for any combination of these values within the corresponding bounds (Lemma 14).

Since it may be difficult to investigate the structure of  $\ker[\omega]$ , we give a weaker lower bound not involving  $h(\ker[\omega])$ :

$$m(\omega) \geq \text{rk } \omega - g - \frac{k(\omega)}{2}$$

(Corollary 12), which can, though, be easier to calculate. This estimate is efficient only for large  $\text{rk } \omega$ , specifically, for  $\text{rk } \omega \geq g$ . However, this is the “majority” of all forms: the forms in general position have  $\text{rk } \omega = 2g$ .

The paper is organized as follows. Section 2 introduces some necessary definitions and facts concerning a Morse form foliation. In Section 3 we prove our main result:  $c(\omega) + m(\omega) = g - \frac{1}{2}k(\omega)$ . Finally, in Section 4 we give the bounds on  $m(\omega)$ .

## 2. DEFINITIONS AND BASIC FACTS

Let us introduce, for future reference, some necessary notions and facts about Morse forms and their foliations.

**2.1. Morse form.** A closed 1-form on  $M$  is called a *Morse form* if it is locally the differential of a Morse function. Let  $\omega$  be a Morse form and  $\text{Sing } \omega = \{p \in M \mid \omega(p) = 0\}$  the set of its singularities; this set is finite since the singularities are isolated and  $M$  is compact.

By the Morse lemma, in a neighborhood of  $p \in \text{Sing } \omega$  on  $M_g^2$  there exist local coordinates  $(x^1, x^2)$  such that  $\omega(x) = \pm x^1 dx^1 + x^2 dx^2$ . If the sign is positive then  $p$  is a *center*, otherwise  $p$  is a *conic singularity*. We denote the set of centers by  $\Omega_0$  and that of conic singularities by  $\Omega_1$ , so that  $\text{Sing } \omega = \Omega_0 \cup \Omega_1$ . By the Poincaré—Hopf theorem, it holds

$$(5) \quad |\Omega_1| - |\Omega_0| = 2g - 2.$$

The *rank* of a closed 1-form  $\omega$  is the rank of its group of periods:

$$\text{rk } \omega = \text{rk}_{\mathbb{Q}} \left\{ \int_{z_1} \omega, \dots, \int_{z_{2g}} \omega \right\},$$

where  $z_1, \dots, z_{2g}$  is a basis of  $H_1(M_g^2)$ . For an exact form,  $\text{rk } \omega = 0$ .

**2.2. Morse form foliation.** On  $M \setminus \text{Sing } \omega$ , the form  $\omega$  defines a foliation  $\mathcal{F}_\omega$ . A leaf  $\gamma \in \mathcal{F}_\omega$  is *compactifiable* if  $\gamma \cup \text{Sing } \omega$  is compact (compact leaves are compactifiable); otherwise it is *non-compactifiable*. If a foliation contains only compactifiable leaves, it is called *compactifiable*.

The foliation  $\mathcal{F}_\omega$  defines a decomposition of  $M$  into mutually disjoint sets [5]; see Figure 2(a),(c) below:

$$(6) \quad M = \left( \bigcup \mathcal{C}_i^{\max} \right) \cup \left( \bigcup \mathcal{C}_j^{\min} \right) \cup \left( \bigcup \overline{\gamma_k^0} \right) \cup \text{Sing } \omega.$$

The *maximal components*  $\mathcal{C}_i^{\max}$  are connected components of the union of all compact leaves. On two-manifolds the notion of maximal component coincides with the notion of periodic component [10]. If  $\text{Sing } \omega \neq \emptyset$ , each maximal component is a cylinder over a compact leaf:  $\mathcal{C}_i^{\max} \cong \gamma_i \times (0, 1)$ . Consider the group  $H_\omega \subseteq H_{n-1}(M)$  generated by the homology classes of all compact leaves;  $H_\omega = \langle [\gamma_i], \gamma_i \in \mathcal{F}_\omega \rangle$  [3]. We denote by  $c(\omega) = \text{rk } H_\omega$  the number of homologically independent compact leaves.

The *minimal components*  $\mathcal{C}_j^{\min}$  of the foliation are connected components of the the set covered by all non-compactifiable leaves. A foliation consisting of exactly one minimal component (and no maximal components) is called *minimal*. Each non-compactifiable leaf is dense in its minimal component [1, 8]. We denote by  $m(\omega)$  the number of minimal components. *Par abus de langage*, we say that a minimal component  $\mathcal{C}^{\min}$  *contains* a leaf or singularity, or the leaf or singularity is *inside* the minimal component, if it belongs to  $\text{int}(\overline{\mathcal{C}^{\min}})$ . We denote by  $k(\omega) = \sum_{i=1}^{m(\omega)} |\text{int}(\overline{\mathcal{C}_i^{\min}}) \cap \text{Sing } \omega|$  the number of singularities inside minimal components; in Figure 5,  $k(\omega) = 2$ .

The components  $\mathcal{C}_i^{\max}$  and  $\mathcal{C}_j^{\min}$  are open; their boundaries lie in the union  $(\bigcup_k \overline{\gamma_k^0}) \cup \text{Sing } \omega$  of non-compact compactifiable leaves and singularities. The number of components, as well as the number of non-compact compactifiable leaves  $\gamma_k^0$ , is finite.

**2.3. Weakly generic Morse form.** While a foliation  $\mathcal{F}_\omega$  is defined on  $M \setminus \text{Sing } \omega$ , a *singular foliation*  $\overline{\mathcal{F}}_\omega$  is defined on the whole  $M$ : two points  $p, q \in M$  belong to the same *leaf* of  $\overline{\mathcal{F}}_\omega$  if there exists a path  $\alpha : [0, 1] \rightarrow M$  with  $\alpha(0) = p$ ,  $\alpha(1) = q$  and  $\omega(\dot{\alpha}(t)) = 0$  for all  $t$  [2]. A *singular leaf* contains a singularity.

On  $M \setminus \text{Sing } \omega$ ,  $\overline{\mathcal{F}}_\omega$  differs from  $\mathcal{F}_\omega$  only by possibly merging together some of its leaves: indeed, non-singular leaves of  $\overline{\mathcal{F}}_\omega$  are leaves of  $\mathcal{F}_\omega$ ; the number of singular leaves of  $\overline{\mathcal{F}}_\omega$  is finite, and each such leaf consists of a finite number of non-compact leaves of  $\mathcal{F}_\omega$  and singularities.

A Morse form is called *generic* if each of its singular leaves contains a unique singularity [2]. On  $M_g^2$  this means that each non-compact compactifiable leaf is compactified by only one singularity. The set of generic forms is dense in any cohomology class [2].

We call a form *weakly generic* if its non-compact compactifiable leaves lying *outside* minimal components are compactified by only one singularity, while those inside minimal components can form segments, as  $\gamma^0$  in Figure 1(a). On  $M \setminus \bigcup_{i=1}^{m(\omega)} \text{int}(\overline{C_i^{min}})$  a weakly generic foliation is generic: all its compact singular leaves are either centers or figures of eight, and connected components of the boundaries of minimal components are single-leaf circles; see Figure 2.

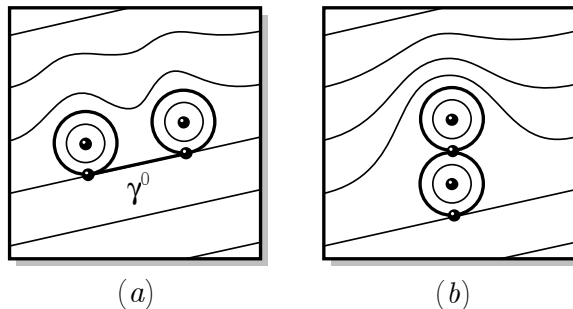


FIGURE 1. Foliations on  $T^2$  with one minimal component. The form (a) is weakly generic, though not generic; the form (b) is not.

**2.4. Foliation graph.** The configuration formed by the *maximal* components in the decomposition (6) is described by the *foliation graph*. Rewrite (6) as

$$M = \left( \bigcup C_i^{max} \right) \cup \left( \bigcup P_j \right),$$

where  $P_j$  are connected components of the union  $P = \left( \bigcup C_j^{min} \right) \cup \left( \bigcup \gamma_k^0 \right) \cup \text{Sing } \omega$  of all non-compact leaves and singularities.

Since  $\partial C_i^{max} \subseteq P$  consists of one or two connected components, each  $C_i^{max}$  adjoins one or two of  $P_j$ . This allows representing  $M$  as a connected graph  $\Gamma$  with edges  $C_i^{max}$  and vertices  $P_j$ : an edge  $C_i^{max}$  is incident to a vertex  $P_j$  if  $\partial C_i^{max} \cap P_j \neq \emptyset$ ; see Figure 2.

We call those vertices  $P_j^I$  that consist solely of compactifiable leaves and singularities I-vertices, see Figure 2(b); II-vertices  $P_j^{II}$  contain minimal components, such as  $P_2$  in Figure 2(d). Note that I-vertices are compact singular leaves (including center singularities). A II-vertex can contain several minimal components separated by compactifiable leaves.

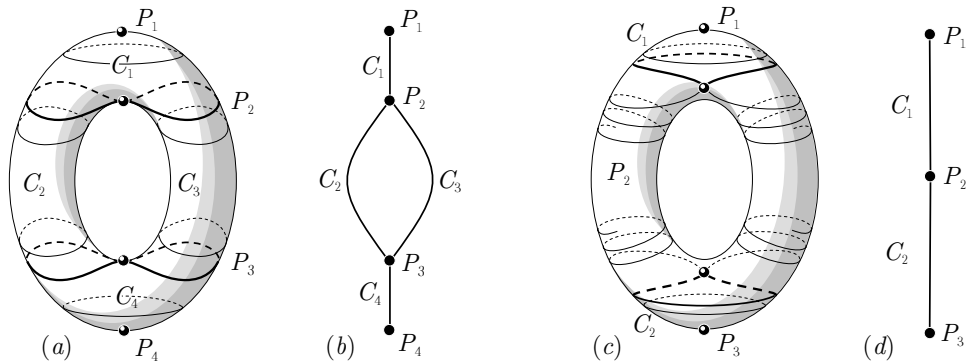


FIGURE 2. (a), (c) Examples of the decomposition (6). (b) Vertices of  $\Gamma$  can include singularities, non-compact compactifiable leaves, and (d) whole minimal components.

3. TOTAL NUMBER OF HOMOLOGICALLY INDEPENDENT COMPACT LEAVES AND MINIMAL COMPONENTS

**Lemma 1.** *Let  $P$  be a I-vertex. Then  $\deg P = 1$  iff  $P$  is a center.*

*Proof.* If  $P$  is a center, in its neighborhood the manifold foliates into circles. Thus a unique cylinder adjoins  $P$ , and so  $\deg P = 1$ .

Conversely, if  $P$  is not a center, then  $P = (\bigcup_i \gamma_i^0) \cup (\bigcup_j s_j)$ , where  $\gamma_i^0$  are non-compact compactifiable leaves and  $s_j \in \Omega_1$ . In a neighborhood of  $P$  the form is exact:  $\omega = df$ ,  $f(P) = 0$ . The components covering the areas  $\{f > 0\}$  and  $\{f < 0\}$  are locally distinct. Since  $P$  is a I-vertex, these have to be maximal components, which means  $\deg P \geq 2$ .  $\square$

**Lemma 2.** *Let  $\gamma^0 \in \mathcal{F}_\omega$  be a non-compact compactifiable leaf such that  $\gamma^0 \cup s$  is compact for some  $s \in \text{Sing } \omega$ . Then in any neighborhood of  $\overline{\gamma^0} = \gamma^0 \cup s$  there exists a compact leaf  $\gamma \in \mathcal{F}_\omega$ .*

*Proof.* Similarly, consider a small cylindrical neighborhood  $U$  of  $\overline{\gamma^0}$  such that  $U \cap \text{Sing } \omega = \{s\}$ . In this neighborhood,  $\omega = df$ ; let  $f(\gamma^0) = 0$ . The set  $U \setminus \overline{\gamma^0}$  has two connected components  $U_1, U_2$ . Locally there are exactly four (non-compact) leaves adjoining  $s$ , and  $f$  changes sign when crossing a leaf. Since  $U \cap \text{Sing } \omega = \{s\}$ , the function  $f$  has a constant sign in one of  $U_i$  (see Figure 3); let  $f > 0$  in  $U_1$ . Then there exists  $t > 0$  such that a connected component  $\gamma$  of  $f^{-1}(t)$  is a compact leaf and lies in  $U$ .  $\square$

The condition of Lemma 2 requires the leaf to be compactified by only one singularity. For leaves compactified by more than one singularity the conclusion of Lemma 2 may not hold: there exist non-compact compactifiable leaves without compact leaves in their neighborhood; see Figure 4.

**Proposition 3.** *Let  $P$  be a I-vertex of a weakly generic form. Then either  $P$  is a center or  $\deg P = 3$ .*

*Proof.* If  $P$  is not a center, then  $P = S^1 \vee_s S^1$ ,  $s \in \Omega_1$ . As in Lemma 2, in a small neighborhood of  $P$  the form is exact, so leaves of the foliation are levels of

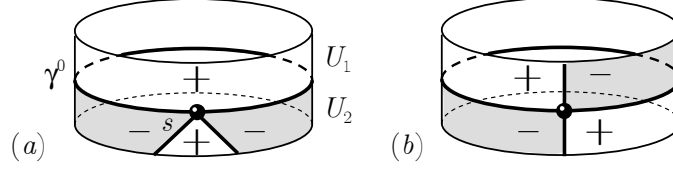


FIGURE 3. Possible (a) and impossible (b) configuration of the leaves adjoining the singularity  $s$ . Areas with different sign of  $f$  are shown in different colors.

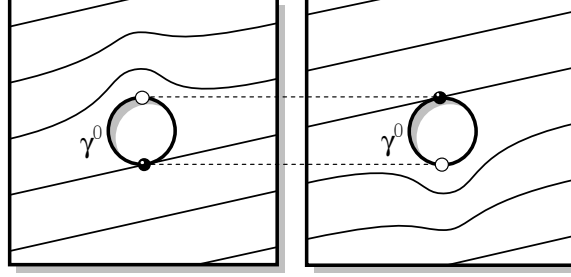


FIGURE 4. Foliation on  $M_2^2 = T^2 \# T^2$  (connected sum) with one compactifiable leaf  $\gamma^0$ , two minimal components, and without compact leaves.

a Morse function. Since  $P$  contains a unique singularity, close levels have one and two connected components, correspondingly. Thus  $\deg P = 3$ .  $\square$

**Proposition 4.** *Let  $P$  be a II-vertex of a weakly generic form. Then*

- (i)  $P$  contains a unique minimal component  $\mathcal{C}^{min}$ ;
- (ii) each connected component of  $\partial\overline{\mathcal{C}^{min}}$  locally attaches to  $\mathcal{C}^{min}$  exactly one maximal component;
- (iii)  $\deg P = |\partial\overline{\mathcal{C}^{min}} \cap \text{Sing } \omega|$ .

*Proof.* Since  $P$  is a II-vertex, it contains a minimal component  $\mathcal{C}^{min}$ . Each connected component  $\partial_i$  of  $\partial\overline{\mathcal{C}^{min}}$  is compact and includes exactly one  $s \in \text{Sing } \omega$ , which adjoins at least one non-compactifiable leaf and at least one non-compact compactifiable leaf  $\gamma^0$ , which adjoins only this singularity. Thus  $\partial_i = \gamma^0 \cup s$ . By Lemma 2, there is exactly one maximal component  $\mathcal{C}_i^{max}$  glued to  $\mathcal{C}^{min}$  by  $\partial_i$ ; see Figure 3(a). Therefore  $P$  consists of  $\overline{\mathcal{C}^{min}}$  with  $|\partial\overline{\mathcal{C}^{min}} \cap \text{Sing } \omega|$  maximal components locally attached to it (globally they can be different ends of the same cylinder).  $\square$

Now we are ready to prove our main theorem:

**Theorem 5.** *Let  $\omega$  be a weakly generic Morse form on  $M_g^2$ . Then*

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2}.$$

*Proof.* Denote by  $n_i$  the number of vertices of degree  $i$  of the foliation graph  $\Gamma$ ;  $n_i = n_i^I + n_i^{II}$ , where  $n_i^I, n_i^{II}$  are the corresponding numbers for I- and II-vertices.

Similarly, denote  $\Omega_1^I$  and  $\Omega_1^{II}$  the sets of conic singularities belonging to the vertices of each type.

Consider  $n_i^I$ . By Lemma 1, it holds  $n_1^I = |\Omega_0|$ ; Proposition 3 gives  $n_3^I = |\Omega_1^I|$  and  $n_i^I = 0$  for  $i \neq 1, 3$ .

Consider  $n_i^{II}$ . By Proposition 4 (i), each II-vertex contains a unique minimal component, so  $\sum_i n_i^{II} = m(\omega)$ . Denote  $k_j = |\text{int}(\overline{C_j^{min}}) \cap \text{Sing } \omega|$ . By Proposition 4 (iii),  $|\Omega_1^{II}| = \sum_i in_i^{II} + \sum_j k_j = \sum_i in_i^{II} + k(\omega)$ .

For the cycle rank  $m(\Gamma) = \frac{1}{2} \sum_i (i-2)n_i + 1$  [7] we have

$$\begin{aligned} 2m(\Gamma) &= -n_1^I + n_3^I + \sum_i in_i^{II} - 2 \sum_i n_i^{II} + 2 \\ &= -|\Omega_0| + |\Omega_1^I| + |\Omega_1^{II}| - k(\omega) - 2m(\omega) + 2. \end{aligned}$$

Since  $m(\Gamma) = c(\omega)$  [5] and by (5), this proves the theorem. □

**Corollary 6.** *For weakly generic forms on  $M_g^2$ ,  $g \neq 0$ , it holds*

$$1 \leq c(\omega) + m(\omega) \leq g;$$

*for a given  $M_g^2$  the bounds are exact and all combinations of  $c(\omega)$  and  $m(\omega)$  within these bounds are possible in the class of generic forms.*

*Proof.* If  $c(\omega) + m(\omega) = 0$  then  $m(\omega) = 0$  and thus  $k(\omega) = 0$ ; Theorem 5 gives  $g = 0$ . That all intermediate values are reached for generic forms was shown in [4]. In particular, on any  $M_g^2$ ,  $g \neq 0$ , there exists a minimal foliation [4], see Figure 5, which shows the exactness of the lower bound; the upper bound is reached on  $\omega = df$ . □

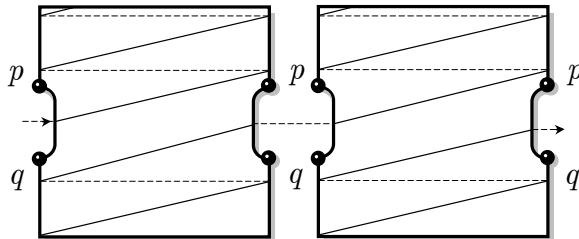


FIGURE 5. Minimal foliation on  $M_2^2 = T^2 \# T^2$ .

The condition for the form to be weakly generic in Corollary 6 is important: on every  $M_g^2$  there exist not weakly generic forms with  $c(\omega) + m(\omega) = 0$ ; see Figure 6.

Theorem 5 and Corollary 6 give:

**Corollary 7.** *For a weakly generic form on  $M_g^2$ ,  $k(\omega)$  is even. In addition,*

$$0 \leq k(\omega) \leq 2g - 2$$

*if  $g \neq 0$ , otherwise  $k(\omega) = 0$ . On a given  $M_g^2$  the bounds are exact and all (even) intermediate values are possible in the class of generic forms.*

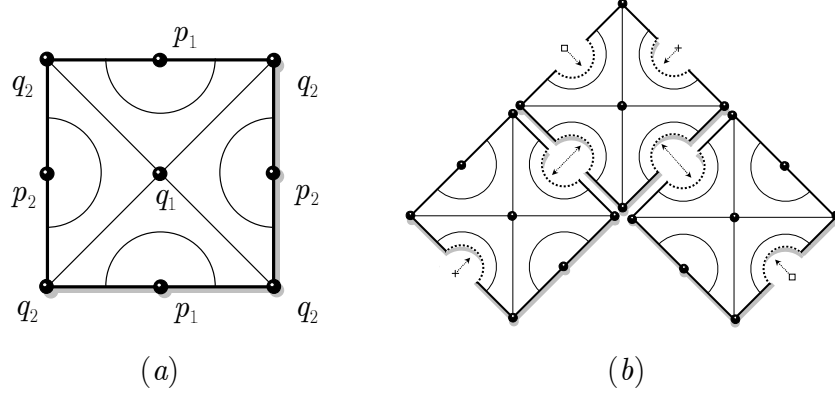


FIGURE 6. Compactifiable foliation with  $c(\omega) = 0$  on (a)  $T^2$ , (b)  $M_g^2 = \#T_i^2$ .

#### 4. BOUNDS ON THE NUMBER OF MINIMAL COMPONENTS

The inequality (1) gives an upper bound on the number of minimal components of a Morse form:  $m(\omega) \leq g$ ; this fact was also proved in [9]. We obtain a lower bound and a better upper bound on  $m(\omega)$  for weakly generic Morse forms.

Consider on  $H_1(M_g^2)$  the intersection of cycles:

$$\cdot : H_1(M_g^2) \times H_1(M_g^2) \rightarrow \mathbb{Z};$$

it is skew-symmetric and non-degenerated. A subgroup  $H \subset H_1(M_g^2)$  is called *isotropic* with respect to the intersection  $\cdot$  if for any  $z, z' \in H$  it holds  $z \cdot z' = 0$  [12]. For an isotropic subgroup,  $\text{rk } H \leq g$ .

For  $G \subseteq H_1(M_g^2)$ , denote  $h(G) = \text{rk } H$ , where  $H \subseteq G$  is a maximal isotropic subgroup. For higher-dimensional manifolds  $M$  this value would depend on the choice of  $H$ ; the maximal rank of an isotropic subgroup is an important topological invariant of a manifold denoted  $h(M)$  [3, 12];  $h(M_g^2) = h(H_1(M)) = g$  [13]. For  $M_g^2$ , though, this definition does not depend on the choice of  $H$ :

**Lemma 8.** *Let  $G \subseteq H_1(M_g^2)$ . Then*

$$h(G) = \text{rk } G - \frac{\text{rk } \|z_i \cdot z_j\|}{2},$$

where  $\{z_i\}$  is a basis of  $G$ .

*Proof.* Obviously,  $\text{rk } \|z_i \cdot z_j\|$  does not depend on the choice of the basis  $\{z_i\}$ . Let  $H \subseteq G$  be a maximal isotropic subgroup; denote  $n = \text{rk } G$ ,  $h = \text{rk } H$ . Choose a basis  $\{z_i\}$  such that  $z_i \in H$  for  $i \leq h$ . Consider  $A = \|z_i \cdot z_j\|$ :

|   |     |   |   |   |
|---|-----|---|---|---|
|   | 1   |   | h |   |
| 0 | ... | 0 |   |   |
| ⋮ |     | ⋮ |   | B |
| 0 | ... | 0 |   |   |
|   | C   |   |   |   |

Since  $H$  is maximal, the  $n - h$  columns of  $B$  are independent, and so are the rows of  $C = -B^T$  and thus some  $n - h$  its columns. The corresponding  $2(n - h)$  columns of  $A$  are independent, and no greater system of columns is independent. Thus  $\text{rk } A = 2(n - h)$ .  $\square$

**Corollary 9.** *It holds*

$$\frac{\text{rk } G}{2} \leq h(G) \leq \min\{\text{rk } G, g\}.$$

Consider the subgroup  $\ker[\omega] = \{z \in H_1(M_g^2) \mid \int_z \omega = 0\}$ ; obviously,  $\text{rk } \ker[\omega] = 2g - \text{rk } \omega$  and thus

$$(7) \quad g - \frac{\text{rk } \omega}{2} \leq h(\ker[\omega]) \leq \min\{2g - \text{rk } \omega, g\}.$$

In particular,

$$(8) \quad 0 \leq h(\ker[\omega]) \leq g.$$

Since  $H_\omega \subseteq \ker[\omega]$ ,

$$(9) \quad c(\omega) \leq h(\ker[\omega]).$$

It can be shown [6] that if  $k(\omega) = 0$  and  $m(\omega) \leq 1$  then

$$(10) \quad h(\ker[\omega]) = c(\omega) = g - m(\omega).$$

A lower bound on  $m(\omega)$  can be given in terms of the structure of  $\ker[\omega]$ . Theorem 5, (9), and (10) give:

**Theorem 10.** *For weakly generic forms  $\omega$  on  $M_g^2$  it holds*

$$(11) \quad g - \frac{k(\omega)}{2} - h(\ker[\omega]) \leq m(\omega) \leq g - \frac{k(\omega)}{2}.$$

*In addition,*

- (i)  $m(\omega) > 0$  if  $k(\omega) > 0$ ;
- (ii)  $m(\omega) \neq 1$  if  $k(\omega) = 0$  and  $h(\ker[\omega]) = g$ .

*On a given  $M_g^2$ , the bounds given by the system (11) and (i) are exact, and all intermediate values are reached except for the case specified in (ii).*

Exactness of the bounds and existence of all intermediate values are shown in Lemma 14 below.

Note that if  $k(\omega) = 0$  then the left side of (11) is non-negative (can be zero) and the bound given by (11) alone is exact. However, if  $k(\omega) > 0$  then the left side of (11) can be zero or even negative and (i) can give a better bound. As an example, consider the foliation in Figure 5, assuming the periods  $(1, \sqrt{2})$  in each torus; then  $h(\ker[\omega]) = 1$  and the left side of (11) is zero. Assuming the periods  $(1, \sqrt{2})$  and  $(1, -\sqrt{2})$ , we have  $h(\ker[\omega]) = 2$  and the left side of (11) negative.

Note also that if  $k(\omega) = 0$  and  $h(\ker[\omega]) = g$ , then  $m(\omega) = 0, 1, 2, 3, \dots, g$ .

**Corollary 11.** *For a weakly generic form on  $M_g^2$ ,  $m(\omega) = 0$  implies  $h(\ker[\omega]) = g$ .*

The converse is not true; a counterexample is a connected sum  $T^2 \# T^2$  with windings with the periods  $(1, \sqrt{2})$  and  $(1, -\sqrt{2})$ , correspondingly.

Since  $H \subseteq \ker[\omega]$  implies  $\text{rk } H \leq 2g - \text{rk } \omega$ ; Theorem 10 gives:

**Corollary 12.** *For weakly generic forms  $\omega$  on  $M_g^2$  it holds*

$$m(\omega) \geq \text{rk } \omega - g - \frac{k(\omega)}{2}.$$

Though this bound is weaker than (11), it is easier to calculate. This bound is efficient for forms with large  $\text{rk } \omega$ , which are the ‘‘majority’’ of all forms: a form in general position has  $\text{rk } \omega = 2g$ . In general case, a Morse form with  $\text{rk } \omega = 2g$  (i.e.,  $\ker[\omega] = 0$ ) has  $c(\omega) = 0$  [5] and  $m(\omega) \geq 1$  [3]. For weakly generic forms, Theorem 10 gives an exact value:

**Corollary 13.** *For weakly generic forms  $\omega$  on  $M_g^2$  such that  $\text{rk } \omega = 2g$ , it holds*

$$m(\omega) = g - \frac{k(\omega)}{2}.$$

Note that for  $c(\omega)$ , (9) and (7) give a bound not involving  $k(\omega)$ :

$$c(\omega) \leq h(\ker[\omega]) \leq 2g - \text{rk } \omega.$$

The following lemma shows that we have given a complete account of the relations between  $g$ ,  $k(\omega)$ ,  $h(\ker[\omega])$ , and  $m(\omega)$ :

**Lemma 14.** *For any  $g \geq 0$ ,  $k$ ,  $m$ , and  $h$  satisfying the constraints of Corollary 7, Theorem 10, and (8), on  $M_g^2$  there exists a generic form  $\omega$  such that  $k(\omega) = k$ ,  $m(\omega) = m$ , and  $h(\ker[\omega]) = h$ .*

*Proof.* Consider  $g$ ,  $k$ ,  $h$ , and  $m$  satisfying the constraints:

$$0 \leq g,$$

$$\text{Corollary 7: } 0 \leq k \leq 2g - 2 \text{ (} k = 0 \text{ if } g = 0\text{); } k \text{ is even,}$$

$$\text{Theorem 10: } 0 \leq m \leq g - \frac{1}{2}k,$$

$$\text{Theorem 10, (8): } c \leq h \leq g; h < g \text{ if } k = 0 \text{ and } m = 1,$$

where  $c = g - \frac{1}{2}k - m$ . If  $g = 0$  then  $k = m = 0$  and the statement trivially holds, so we assume  $g > 0$ . In the rest of the proof we assume that all unspecified periods of  $\omega$  are incommensurable.

Let  $k = 0$  and  $m \leq 1$ ; then  $h = c$ . An example is a connected sum  $\sharp_{j=1}^c T_j$  of tori with a compact foliation each plus, if  $m = 1$ , a torus with a minimal foliation. By (10),  $h(\ker[\omega]) = h$ .

Let  $k = 0$  and  $2 \leq m \leq g$ . Consider a connected sum  $\sharp$  of  $m$  tori  $T_i^{(m)}$  with a minimal foliation and  $c = g - m$  tori  $T_j^{(c)}$  with a compact foliation. Complete  $H_\omega$  to a maximal isotropic subgroup  $H \subseteq \ker[\omega]$  such that  $\text{rk } H = h$ . Namely, denote  $h^{(m)} = h - c$ ; obviously,  $0 \leq h^{(m)} \leq m$ .

- (i) Let  $h^{(m)} = 0$ . Then just choose all incommensurable periods in all  $T_i^{(m)}$ .
- (ii) Let  $h^{(m)} = 1$ . Choose the periods  $(1, \sqrt{2})$  in  $T_1^{(m)}$  and  $(1, \sqrt{3})$  in  $T_2^{(m)}$ . Then  $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{21} \rangle$ , where  $z_{i1}, z_{i2}$  are the basic cycles of  $T_i^{(m)}$  corresponding to these periods.
- (iii) Let  $h^{(m)} = 2$ . Similarly, choose the periods  $(1, \sqrt{2})$  and  $(\sqrt{2}, 1)$  in the first two  $T_i^{(m)}$ . Then  $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{22}, z_{12} - z_{21} \rangle$  is isotropic.

- (iv) Let  $h^{(m)} = 3$ . Choose the periods  $(1, \sqrt{2})$ ,  $(\sqrt{2}, -1)$ , and  $(\sqrt{2} - 1, 2\sqrt{2})$  in the first three  $T_i^{(m)}$ . By Lemma 8, the isotropic subgroup  $\langle z_{11} - z_{21} + z_{31}, z_{12} + z_{22} - z_{31}, z_{12} + z_{21} - z_{32} \rangle$  of  $\ker[\omega|_{\sharp T_i^{(m)}}]$  is maximal.
  - (v) Let  $h^{(m)} = 2n$ ,  $n \in \mathbb{N}$ . Consider  $n$  pairs of tori with periods  $(\alpha_i, \alpha_i\sqrt{2})$  and  $(\alpha_i\sqrt{2}, \alpha_i)$ , so that each pair behaves as in (iii) above, but different pairs are incommensurable.
  - (vi) Let  $h^{(m)} = 2n + 1$ . Choose  $n - 1$  pairs as in (v) and a triple as in (iv).
- By construction, we obtain  $h(\ker[\omega]) = c + h^{(m)} = h$ .

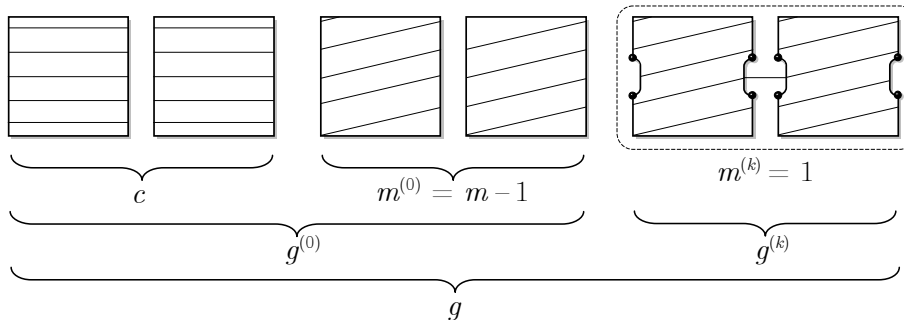


FIGURE 7. Construction of the foliation in Lemma 14.

Let now  $k \geq 2$ , thus  $g \geq \frac{1}{2}k + 1$ . Construct a manifold  $M^{(k)}$  with  $g^{(k)} = \frac{1}{2}k + 1$ ,  $m(\omega^{(k)}) = 1$ ,  $k(\omega^{(k)}) = k$  as shown in Figure 5 and a manifold  $M^{(0)}$  with  $g^{(0)} = g - g^{(k)}$ ,  $m(\omega^{(0)}) = m - 1$ ,  $k(\omega^{(0)}) = 0$  as discussed above; see Figure 7. Then  $M^{(k)} \sharp M^{(0)}$  has the desired properties. To obtain  $h(\ker[\omega]) = h$ ,  $M^{(0)}$  is to be constructed with  $h^{(0)} = \min(h, g^{(0)})$  and in  $M^{(k)}$ , the periods are constructed as in (i)–(vi) above with  $h^{(k)} = h - h^{(0)}$  if positive.  $\square$

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