THE NUMBER OF MINIMAL COMPONENTS AND HOMOLOGICALLY INDEPENDENT COMPACT LEAVES OF A WEAKLY GENERIC MORSE FORM ON A CLOSED SURFACE

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Abstract. On a closed orientable surface M_g^2 of genus g, we consider the foliation of a weakly generic Morse form ω on M_g^2 and show that for such forms $c(\omega) + m(\omega) = g - \frac{1}{2}k(\omega)$, where $c(\omega)$ is the number of homologically independent compact leaves of the foliation, $m(\omega)$ is the number of its minimal components, and $k(\omega)$ is the total number of singularities of ω that are surrounded by a minimal component. We also give lower bounds on $m(\omega)$ in terms of $k(\omega)$ and the form rank $\operatorname{rk}\omega$ or the structure of $\ker[\omega]$, where $[\omega]$ is the integration map.

1. Introduction

Consider a closed connected orientable smooth two-dimensional manifold $M = M_g^2$ of genus g. Let ω be a Morse form on M, i.e., a closed 1-form with Morse singularities $\operatorname{Sing}\omega$, locally the differential of a Morse function. This form defines a foliation $\mathcal{F}_\omega$ on $M \setminus \operatorname{Sing}\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is called compactifiable if $\gamma \cup \operatorname{Sing}\omega$ is compact.

A Morse form is called *generic* if each of its non-compact compactifiable leaves is compactified by a unique singularity [2, Definition 9.1]. The set of such forms is dense in any cohomology class [2, Lemma 9.2]. The term *generic* introduced in [2] is somewhat misleading because the set of such forms is not open. We find it plausible that such forms are the "majority" of Morse forms and thus their properties are in a sense "typical," though we are not aware of any proof of this.

Our results hold for a wider class of forms, which we call *weakly generic*: the requirement for a leaf to be compactified by only one singularity is only applied to the leaves not surrounded by minimal components.

The number $m(\omega)$ of minimal components and $c(\omega)$ of homologically independent compact leaves are important topological characteristics of the foliation. On M_g^2 it holds [5]

$$0 \le c(\omega) + m(\omega) \le g \tag{1}$$

and all such combinations are possible on a given M [4]. In particular, if $c(\omega) = g$ then the foliation is compactifiable, i.e., $m(\omega) = 0$, though the converse is not true: there exist compactifiable foliations with $c(\omega) < g$.

In this paper, for weakly generic forms we give a precise expression for $c(\omega) + m(\omega)$ and better bounds on $m(\omega)$. A useful characteristic of a weakly generic form foliation is the number $k(\omega)$ of singularities that are surrounded by a minimal

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component; for a weakly generic form $k(\omega)$ is even (Corollary 7). Our main result states that for such forms the inequality (1) becomes

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2} \tag{2}$$

(Theorem 5). In particular, for weakly generic forms on M^2_g, $g \neq 0$, the exact lower bound in (1) is

$$1 \leq c(\omega) + m(\omega) \leq g$$

(Corollary 6). On the other hand, (2) gives a criterion for compactifiability for weakly generic forms [11]: $m(\omega) = 0$ iff $c(\omega) = g$.

The inequality (1) gives an upper bound on the number of minimal components: $m(\omega) \leq g$; this was also proved in [9]. For weakly generic forms, (2) gives a better upper bound:

$$m(\omega) \leq g - \frac{k(\omega)}{2}. \tag{3}$$

We are not aware, though, of any *lower* bound on $m(\omega)$ given in literature, except for that if $\operatorname{rk}\omega > g$ (the rank of the group of periods) then the foliation has minimal components: $m(\omega) > 0$ [11]. For weakly generic forms, we give a lower bound on $m(\omega)$, cf. (3):

$$m(\omega) \geq g - \frac{k(\omega)}{2} - h(\ker[\omega]) \tag{4}$$

(Theorem 10). Here, $\ker[\omega] = \langle z \in H_1(M) \mid \int_z \omega = 0 \rangle$ and $h(*)$ is the rank of a maximal subgroup consisting of non-intersecting cycles. We calculate the value of $h(\ker[\omega])$ (Lemma 8) and bound it in terms of $\operatorname{rk}\ker[\omega]$ (Corollary 9).

The bound (4) is not exact; however, it becomes exact together with a trivial observation that $m(\omega) > 0$ if $k(\omega) > 0$. All intermediate values are also reached, except for $m = 1$ when $k = 0$ and $h(\ker[\omega]) = g$; this combination is impossible [6]. Our account of the relationships between g, $k(\omega)$, $h(\ker[\omega])$, and $m(\omega)$ is complete: we build a (generic) form for any combination of these values within the corresponding bounds (Lemma 14).

Since it may be difficult to investigate the structure of $\ker[\omega]$, we give a weaker lower bound not involving $h(\ker[\omega])$:

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2}$$

(Corollary 12), which can, though, be easier to calculate. This estimate is efficient only for large $\operatorname{rk}\omega$, specifically, for $\operatorname{rk}\omega \geq g$. However, this is the "majority" of all forms: the forms in general position have $\operatorname{rk}\omega = 2g$.

The paper is organized as follows. Section 2 introduces some necessary definitions and facts concerning a Morse form foliation. In Section 3 we prove our main result: $c(\omega) + m(\omega) = g - \frac{1}{2}k(\omega)$. Finally, in Section 4 we give the bounds on $m(\omega)$.

2. Definitions and basic facts

Let us introduce, for future reference, some necessary notions and facts about Morse forms and their foliations.

2.1. **Morse form.** A closed 1-form on M is called a *Morse form* if it is locally the differential of a Morse function. Let ω be a Morse form and $\operatorname{Sing}\omega = \{p \in M \mid \omega(p) = 0\}$ the set of its singularities; this set is finite since the singularities are isolated and M is compact.

By the Morse lemma, in a neighborhood of $p \in \operatorname{Sing}\omega$ on M_g^2 there exist local coordinates (x^1, x^2) such that $\omega(x) = \pm x^1 dx^1 + x^2 dx^2$. If the sign is positive then p is a *center*, otherwise p is a *conic singularity.* We denote the set of centers by Ω_0 and that of conic singularities by Ω_1, so that $\operatorname{Sing}\omega = \Omega_0 \cup \Omega_1$. By the Poincaré—Hopf theorem, it holds

$$|\Omega_1| - |\Omega_0| = 2g - 2. \tag{5}$$

The *rank* of a closed 1-form ω is the rank of its group of periods:

$$\operatorname{rk}\omega = \operatorname{rk}_{\mathbb{Q}}\left\{\int_{z_1}\omega, \dots, \int_{z_{2g}}\omega\right\},$$

where $z_1, \dots, z_{2g}$ is a basis of $H_1(M_g^2)$. For an exact form, $\operatorname{rk}\omega = 0$.

2.2. **Morse form foliation.** On $M \setminus \operatorname{Sing}\omega$, the form ω defines a foliation $\mathcal{F}_\omega$. A leaf $\gamma \in \mathcal{F}_\omega$ is *compactifiable* if $\gamma \cup \operatorname{Sing}\omega$ is compact (compact leaves are compactifiable); otherwise it is *non-compactifiable.* If a foliation contains only compactifiable leaves, it is called *compactifiable.*

The foliation $\mathcal{F}_\omega$ defines a decomposition of M into mutually disjoint sets [5]; see Figure 2(a),(c) below:

$$M = \left(\bigcup \mathcal{C}_i^{max}\right) \cup \left(\bigcup \mathcal{C}_j^{min}\right) \cup \left(\bigcup \gamma_k^0\right) \cup \operatorname{Sing}\omega. \tag{6}$$

The *maximal components* $\mathcal{C}_i^{max}$ are connected components of the union of all compact leaves. On two-manifolds the notion of maximal component coincides with the notion of periodic component [10]. If $\operatorname{Sing}\omega \neq \emptyset$, each maximal component is a cylinder over a compact leaf: $\mathcal{C}_i^{max} \cong \gamma_i \times (0,1)$. Consider the group $H_\omega \subseteq H_{n-1}(M)$ generated by the homology classes of all compact leaves; $H_\omega = \langle [\gamma_i], \gamma_i \in \mathcal{F}_\omega\rangle$ [3]. We denote by $c(\omega) = \operatorname{rk} H_\omega$ the number of homologically independent compact leaves.

The *minimal components* $\mathcal{C}_j^{min}$ of the foliation are connected components of the the set covered by all non-compactifiable leaves. A foliation consisting of exactly one minimal component (and no maximal components) is called *minimal.* Each non-compactifiable leaf is dense in its minimal component [1, 8]. We denote by $m(\omega)$ the number of minimal components. *Par abus de langage*, we say that a minimal component $\mathcal{C}^{min}$ *contains* a leaf or singularity, or the leaf or singularity is *inside* the minimal component, if it belongs to $\operatorname{int}(\overline{\mathcal{C}^{min}})$. We denote by $k(\omega) = \sum_{i=1}^{m(\omega)} \left|\operatorname{int}(\overline{\mathcal{C}_i^{min}}) \cap \operatorname{Sing}\omega\right|$ the number of singularities inside minimal components; in Figure 5, $k(\omega) = 2$.

The components $\mathcal{C}_i^{max}$ and $\mathcal{C}_j^{min}$ are open; their boundaries lie in the union $(\bigcup_k \gamma_k^0) \cup \operatorname{Sing}\omega$ of non-compact compactifiable leaves and singularities. The number of components, as well as the number of non-compact compactifiable leaves γ_k^0, is finite.

2.3. **Weakly generic Morse form.** While a foliation $\mathcal{F}_\omega$ is defined on $M \setminus \operatorname{Sing}\omega$, a *singular foliation* $\overline{\mathcal{F}}_\omega$ is defined on the whole M: two points $p, q \in M$ belong to the same *leaf* of $\overline{\mathcal{F}}_\omega$ if there exists a path $\alpha : [0,1] \to M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega(\dot\alpha(t)) = 0$ for all t [2]. A *singular leaf* contains a singularity.

On $M \setminus \operatorname{Sing}\omega$, $\overline{\mathcal{F}}_\omega$ differs from $\mathcal{F}_\omega$ only by possibly merging together some of its leaves: indeed, non-singular leaves of $\overline{\mathcal{F}}_\omega$ are leaves of $\mathcal{F}_\omega$; the number of singular leaves of $\overline{\mathcal{F}}_\omega$ is finite, and each such leaf consists of a finite number of non-compact leaves of $\mathcal{F}_\omega$ and singularities.

A Morse form is called *generic* if each of its singular leaves contains a unique singularity [2]. On M_g^2 this means that each non-compact compactifiable leaf is compactified by only one singularity. The set of generic forms is dense in any cohomology class [2].

We call a form *weakly generic* if its non-compact compactifiable leaves lying *outside* minimal components are compactified by only one singularity, while those inside minimal components can form segments, as γ^0 in Figure 1(a). On $M \setminus \bigcup_{i=1}^{m(\omega)} \operatorname{int}(\overline{\mathcal{C}_i^{min}})$ a weakly generic foliation is generic: all its compact singular leaves are either centers or figures of eight, and connected components of the boundaries of minimal components are single-leaf circles; see Figure 2.

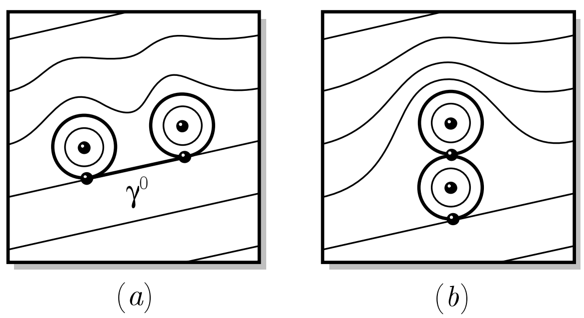

FIGURE 1. Foliations on T^2 with one minimal component. The form (a) is weakly generic, though not generic; the form (b) is not.

2.4. **Foliation graph.** The configuration formed by the *maximal* components in the decomposition (6) is described by the *foliation graph*. Rewrite (6) as

$$M = \left(\bigcup \mathcal{C}_i^{max}\right) \cup \left(\bigcup P_j\right),$$

where P_j are connected components of the union $P = \left(\bigcup \mathcal{C}_j^{min}\right) \cup \left(\bigcup \gamma_k^0\right) \cup \operatorname{Sing}\omega$ of all non-compact leaves and singularities.

Since $\partial\mathcal{C}_i^{max} \subseteq P$ consists of one or two connected components, each $\mathcal{C}_i^{max}$ adjoins one or two of P_j. This allows representing M as a connected graph Γ with edges $\mathcal{C}_i^{max}$ and vertices P_j: an edge $\mathcal{C}_i^{max}$ is incident to a vertex P_j if $\partial\mathcal{C}_i^{max} \cap P_j \neq \emptyset$; see Figure 2.

We call those vertices P_j^I that consist solely of compactifiable leaves and singularities I-vertices, see Figure 2(b); II-vertices P_j^{II} contain minimal components, such as P_2 in Figure 2(d). Note that I-vertices are compact singular leaves (including center singularities). A II-vertex can contain several minimal components separated by compactifiable leaves.

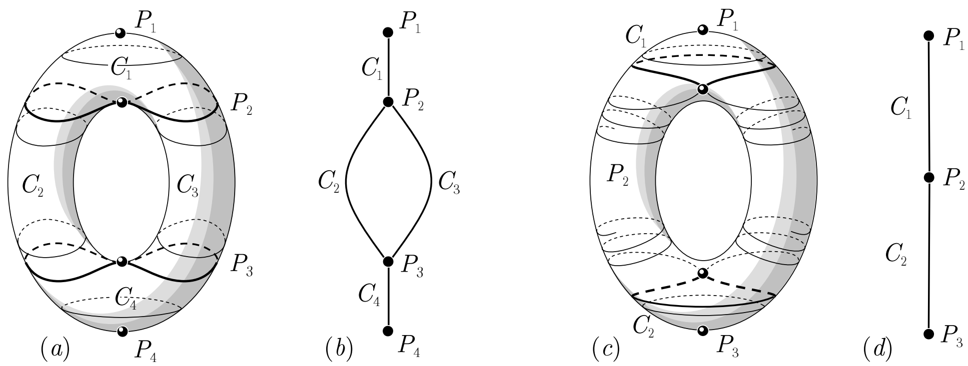

FIGURE 2. $(a), (c)$ Examples of the decomposition (6). (b) Vertices of Γ can include singularities, non-compact compactifiable leaves, and (d) whole minimal components.

3. Total number of homologically independent compact leaves and minimal components

Lemma 1. *Let P be a I-vertex. Then* $\deg P = 1$ *iff P is a center.*

Proof. If P is a center, in its neighborhood the manifold foliates into circles. Thus a unique cylinder adjoins P, and so $\deg P = 1$.

Conversely, if P is not a center, then $P = (\bigcup_i \gamma_i^0) \cup (\bigcup_j s_j)$, where γ_i^0 are non-compact compactifiable leaves and $s_j \in \Omega_1$. In a neighborhood of P the form is exact: $\omega = df$, $f(P) = 0$. The components covering the areas $\{f > 0\}$ and $\{f < 0\}$ are locally distinct. Since P is a I-vertex, these have to be maximal components, which means $\deg P \geq 2$. □

Lemma 2. *Let $\gamma^0 \in \mathcal{F}_\omega$ be a non-compact compactifiable leaf such that $\gamma^0 \cup s$ is compact for some $s \in \operatorname{Sing}\omega$. Then in any neighborhood of $\overline{\gamma^0} = \gamma^0 \cup s$ there exists a compact leaf $\gamma \in \mathcal{F}_\omega$.*

Proof. Similarly, consider a small cylindrical neighborhood U of $\overline{\gamma^0}$ such that $U \cap \operatorname{Sing}\omega = \{s\}$. In this neighborhood, $\omega = df$; let $f(\gamma^0) = 0$. The set $U \setminus \overline{\gamma^0}$ has two connected components U_1, U_2. Locally there are exactly four (non-compact) leaves adjoining s, and f changes sign when crossing a leaf. Since $U \cap \operatorname{Sing}\omega = \{s\}$, the function f has a constant sign in one of U_i (see Figure 3); let $f > 0$ in U_1. Then there exists $t > 0$ such that a connected component γ of $f^{-1}(t)$ is a compact leaf and lies in U. □

The condition of Lemma 2 requires the leaf to be compactified by only one singularity. For leaves compactified by more than one singularity the conclusion of Lemma 2 may not hold: there exist non-compact compactifiable leaves without compact leaves in their neighborhood; see Figure 4.

Proposition 3. *Let P be a I-vertex of a weakly generic form. Then either P is a center or* $\deg P = 3$.

Proof. If P is not a center, then $P = S^1 \vee_s S^1$, $s \in \Omega_1$. As in Lemma 2, in a small neighborhood of P the form is exact, so leaves of the foliation are levels of

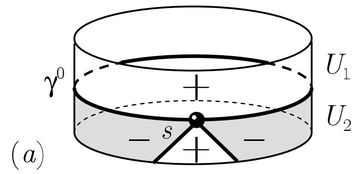

FIGURE 3. Possible (a) and impossible (b) configuration of the leaves adjoining the singularity s. Areas with different sign of f are shown in different colors.

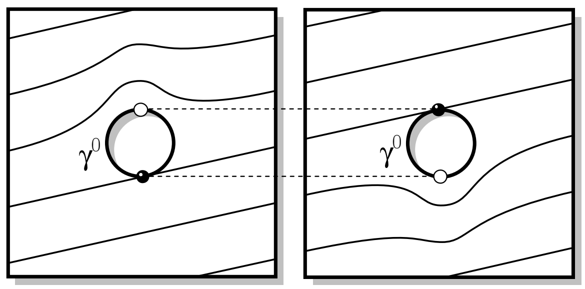

FIGURE 4. Foliation on $M_2^2 = T^2 \sharp T^2$ (connected sum) with one compactifiable leaf γ^0, two minimal components, and without compact leaves.

a Morse function. Since P contains a unique singularity, close levels have one and two connected components, correspondingly. Thus $\deg P = 3$. □

Proposition 4. *Let P be a II-vertex of a weakly generic form. Then*

(i) *P contains a unique minimal component $\mathcal{C}^{min}$;*
(ii) *each connected component of $\partial\overline{\mathcal{C}^{min}}$ locally attaches to $\mathcal{C}^{min}$ exactly one maximal component;*
(iii) $\deg P = |\partial\overline{\mathcal{C}^{min}} \cap \operatorname{Sing}\omega|$.

Proof. Since P is a II-vertex, it contains a minimal component $\mathcal{C}^{min}$. Each connected component ∂_i of $\partial\overline{\mathcal{C}^{min}}$ is compact and includes exactly one $s \in \operatorname{Sing}\omega$, which adjoins at least one non-compactifiable leaf and at least one non-compact compactifiable leaf γ^0, which adjoins only this singularity. Thus $\partial_i = \gamma^0 \cup s$. By Lemma 2, there is exactly one maximal component $\overline{\mathcal{C}_i^{max}}$ glued to $\mathcal{C}^{min}$ by ∂_i; see Figure 3(a). Therefore P consists of $\overline{\mathcal{C}^{min}}$ with $|\partial\overline{\mathcal{C}^{min}} \cap \operatorname{Sing}\omega|$ maximal components locally attached to it (globally they can be different ends of the same cylinder). □

Now we are ready to prove our main theorem:

Theorem 5. *Let ω be a weakly generic Morse form on M_g^2. Then*

$$c(\omega) + m(\omega) = g - \frac{k(\omega)}{2}.$$

Proof. Denote by n_i the number of vertices of degree i of the foliation graph Γ; $n_i = n_i^I + n_i^{II}$, where n_i^I, n_i^{II} are the corresponding numbers for I- and II-vertices.

Similarly, denote Ω_1^I and Ω_1^{II} the sets of conic singularities belonging to the vertices of each type.

Consider n_i^I. By Lemma 1, it holds $n_1^I = |\Omega_0|$; Proposition 3 gives $n_3^I = |\Omega_1^I|$ and $n_i^I = 0$ for $i \neq 1, 3$.

Consider n_i^{II}. By Proposition 4 (i), each II-vertex contains a unique minimal component, so $\sum_i n_i^{II} = m(\omega)$. Denote $k_j = \left|\operatorname{int}(\overline{\mathcal{C}_j^{min}}) \cap \operatorname{Sing}\omega\right|$. By Proposition 4 (iii), $|\Omega_1^{II}| = \sum_i i n_i^{II} + \sum_j k_j = \sum_i i n_i^{II} + k(\omega)$.

For the cycle rank $m(\Gamma) = \frac{1}{2}\sum_i (i-2)n_i + 1$ [7] we have

$$\begin{aligned} 2m(\Gamma) &= -n_1^I + n_3^I + \sum_i i n_i^{II} - 2\sum_i n_i^{II} + 2 \\ &= -|\Omega_0| + |\Omega_1^I| + |\Omega_1^{II}| - k(\omega) - 2m(\omega) + 2. \end{aligned}$$

Since $m(\Gamma) = c(\omega)$ [5] and by (5), this proves the theorem. □

Corollary 6. *For weakly generic forms on M_g^2, $g \neq 0$, it holds*

$$1 \leq c(\omega) + m(\omega) \leq g;$$

for a given M_g^2 the bounds are exact and all combinations of $c(\omega)$ and $m(\omega)$ within these bounds are possible in the class of generic forms.

Proof. If $c(\omega) + m(\omega) = 0$ then $m(\omega) = 0$ and thus $k(\omega) = 0$; Theorem 5 gives $g = 0$. That all intermediate values are reached for generic forms was shown in [4]. In particular, on any M_g^2, $g \neq 0$, there exists a minimal foliation [4], see Figure 5, which shows the exactness of the lower bound; the upper bound is reached on $\omega = df$. □

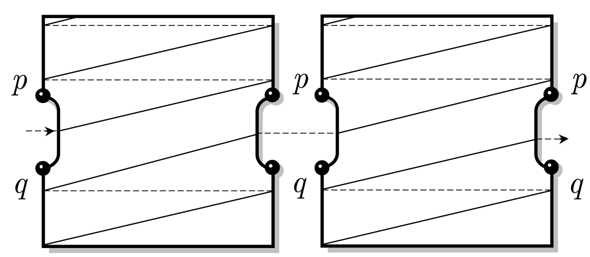

FIGURE 5. Minimal foliation on $M_2^2 = T^2 \sharp T^2$.

The condition for the form to be weakly generic in Corollary 6 is important: on every M_g^2 there exist not weakly generic forms with $c(\omega) + m(\omega) = 0$; see Figure 6.

Theorem 5 and Corollary 6 give:

Corollary 7. *For a weakly generic form on M_g^2, $k(\omega)$ is even. In addition,*

$$0 \leq k(\omega) \leq 2g - 2$$

if $g \neq 0$, otherwise $k(\omega) = 0$. On a given M_g^2 the bounds are exact and all (even) intermediate values are possible in the class of generic forms.

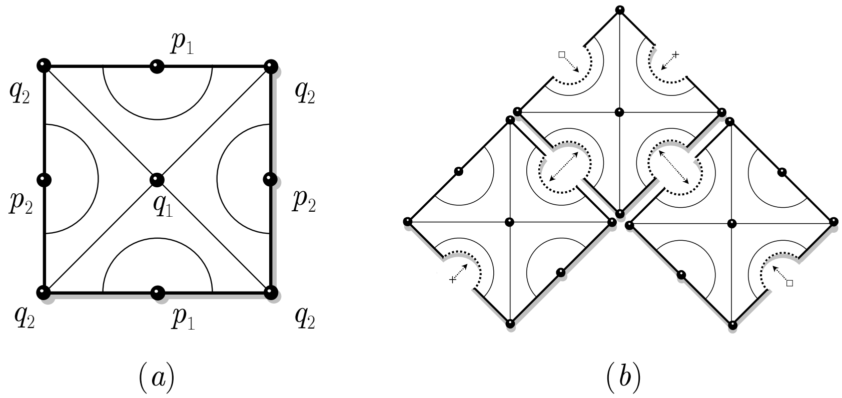

FIGURE 6. Compactifiable foliation with $c(\omega) = 0$ on (a) T^2, (b) $M_g^2 = \sharp\, T_i^2$.

4. Bounds on the number of minimal components

The inequality (1) gives an upper bound on the number of minimal components of a Morse form: $m(\omega) \le g$; this fact was also proved in [9]. We obtain a lower bound and a better upper bound on $m(\omega)$ for weakly generic Morse forms.

Consider on $H_1(M_g^2)$ the intersection of cycles:

$$\cdot : H_1(M_g^2) \times H_1(M_g^2) \to \mathbb{Z};$$

it is skew-symmetric and non-degenerated. A subgroup $H \subset H_1(M_g^2)$ is called *isotropic* with respect to the intersection $\cdot$ if for any $z, z' \in H$ it holds $z \cdot z' = 0$ [12]. For an isotropic subgroup, $\operatorname{rk} H \le g$.

For $G \subseteq H_1(M_g^2)$, denote $h(G) = \operatorname{rk} H$, where $H \subseteq G$ is a maximal isotropic subgroup. For higher-dimensional manifolds M this value would depend on the choice of H; the maximal rank of an isotropic subgroup is an important topological invariant of a manifold denoted $h(M)$ [3, 12]; $h(M_g^2) = h(H_1(M)) = g$ [13]. For M_g^2, though, this definition does not depend on the choice of H:

Lemma 8. *Let $G \subseteq H_1(M_g^2)$. Then*

$$h(G) = \operatorname{rk} G - \frac{\operatorname{rk} \|z_i \cdot z_j\|}{2},$$

where $\{z_i\}$ is a basis of G.

Proof. Obviously, $\operatorname{rk} \|z_i \cdot z_j\|$ does not depend on the choice of the basis $\{z_i\}$. Let $H \subseteq G$ be a maximal isotropic subgroup; denote $n = \operatorname{rk} G$, $h = \operatorname{rk} H$. Choose a basis $\{z_i\}$ such that $z_i \in H$ for $i \le h$. Consider $A = \|z_i \cdot z_j\|$:

$$\begin{array}{c|ccc|c|} \multicolumn{1}{c}{} & \multicolumn{1}{c}{\scriptstyle 1} & & \multicolumn{1}{c}{\scriptstyle h} & \multicolumn{1}{c}{} \\ \cline{2-5} & 0 & \cdots & 0 & \\ & \vdots & & \vdots & B \\ & 0 & \cdots & 0 & \\ \cline{2-5} & & C & & \\ \cline{2-5} \end{array}$$

Since H is maximal, the $n-h$ columns of B are independent, and so are the rows of $C=-B^T$ and thus some $n-h$ its columns. The corresponding $2(n-h)$ columns of A are independent, and no greater system of columns is independent. Thus $\operatorname{rk} A = 2(n-h)$. □

Corollary 9. *It holds*

$$\frac{\operatorname{rk} G}{2} \le h(G) \le \min\{\operatorname{rk} G, g\}.$$

Consider the subgroup $\ker[\omega]=\{z\in H_1(M_g^2) \mid \int_z \omega = 0\}$; obviously, $\operatorname{rk}\ker[\omega] = 2g-\operatorname{rk}\omega$ and thus

$$g-\frac{\operatorname{rk}\omega}{2} \le h(\ker[\omega]) \le \min\{2g-\operatorname{rk}\omega, g\}. \tag{7}$$

In particular,

$$0 \le h(\ker[\omega]) \le g. \tag{8}$$

Since $H_\omega \subseteq \ker[\omega]$,

$$c(\omega) \le h(\ker[\omega]). \tag{9}$$

It can be shown [6] that if $k(\omega)=0$ and $m(\omega)\le 1$ then

$$h(\ker[\omega]) = c(\omega) = g-m(\omega). \tag{10}$$

A lower bound on $m(\omega)$ can be given in terms of the structure of $\ker[\omega]$. Theorem 5, (9), and (10) give:

Theorem 10. *For weakly generic forms ω on M_g^2 it holds*

$$g-\frac{k(\omega)}{2} - h(\ker[\omega]) \le m(\omega) \le g-\frac{k(\omega)}{2}. \tag{11}$$

In addition,

(i) *$m(\omega)>0$ if $k(\omega)>0$;*
(ii) *$m(\omega)\neq 1$ if $k(\omega)=0$ and $h(\ker[\omega])=g$.*

On a given M_g^2, the bounds given by the system (11) *and (i) are exact, and all intermediate values are reached except for the case specified in (ii).*

Exactness of the bounds and existence of all intermediate values are shown in Lemma 14 below.

Note that if $k(\omega)=0$ then the left side of (11) is non-negative (can be zero) and the bound given by (11) alone is exact. However, if $k(\omega)>0$ then the left side of (11) can be zero or even negative and (i) can give a better bound. As an example, consider the foliation in Figure 5, assuming the periods $(1,\sqrt{2})$ in each torus; then $h(\ker[\omega])=1$ and the left side of (11) is zero. Assuming the periods $(1,\sqrt{2})$ and $(1,-\sqrt{2})$, we have $h(\ker[\omega])=2$ and the left side of (11) negative.

Note also that if $k(\omega)=0$ and $h(\ker[\omega])=g$, then $m(\omega)=0,\not{1},2,3,\dots,g$.

Corollary 11. *For a weakly generic form on M_g^2, $m(\omega)=0$ implies $h(\ker[\omega])=g$.*

The converse is not true; a counterexample is a connected sum $T^2\sharp T^2$ with windings with the periods $(1,\sqrt{2})$ and $(1,-\sqrt{2})$, correspondingly.

Since $H\subseteq\ker[\omega]$ implies $\operatorname{rk} H\le 2g-\operatorname{rk}\omega$; Theorem 10 gives:

Corollary 12. *For weakly generic forms* ω *on* M_g^2 *it holds*

$$m(\omega) \geq \operatorname{rk}\omega - g - \frac{k(\omega)}{2}.$$

Though this bound is weaker than (11), it is easier to calculate. This bound is efficient for forms with large $\operatorname{rk}\omega$, which are the "majority" of all forms: a form in general position has $\operatorname{rk}\omega = 2g$. In general case, a Morse form with $\operatorname{rk}\omega = 2g$ (i.e., $\ker[\omega] = 0$) has $c(\omega) = 0$ [5] and $m(\omega) \geq 1$ [3]. For weakly generic forms, Theorem 10 gives an exact value:

Corollary 13. *For weakly generic forms* ω *on* M_g^2 *such that* $\operatorname{rk}\omega = 2g$*, it holds*

$$m(\omega) = g - \frac{k(\omega)}{2}.$$

Note that for $c(\omega)$, (9) and (7) give a bound not involving $k(\omega)$:

$$c(\omega) \leq h(\ker[\omega]) \leq 2g - \operatorname{rk}\omega.$$

The following lemma shows that we have given a complete account of the relations between g, $k(\omega)$, $h(\ker[\omega])$, and $m(\omega)$:

Lemma 14. *For any* $g \geq 0$*,* k*,* m*, and* h *satisfying the constraints of Corollary 7, Theorem 10, and* (8)*, on* M_g^2 *there exists a generic form* ω *such that* $k(\omega) = k$*,* $m(\omega) = m$*, and* $h(\ker[\omega]) = h$*.*

Proof. Consider g, k, h, and m satisfying the constraints:

$$\begin{aligned} & 0 \leq\ g, \\ \text{Corollary 7:}\quad & 0 \leq\ k\ \leq 2g-2\ (k=0 \text{ if } g=0);\ k \text{ is even}, \\ \text{Theorem 10:}\quad & 0 \leq m \leq g - \tfrac{1}{2}k, \\ \text{Theorem 10, (8):}\quad & c \leq\ h\ \leq g;\ h < g \text{ if } k=0 \text{ and } m=1, \end{aligned}$$

where $c = g - \frac{1}{2}k - m$. If $g = 0$ then $k = m = 0$ and the statement trivially holds, so we assume $g > 0$. In the rest of the proof we assume that all unspecified periods of ω are incommensurable.

Let $k = 0$ and $m \leq 1$; then $h = c$. An example is a connected sum $\sharp_{j=1}^{c} T_j$ of tori with a compact foliation each plus, if $m = 1$, a torus with a minimal foliation. By (10), $h(\ker[\omega]) = h$.

Let $k = 0$ and $2 \leq m \leq g$. Consider a connected sum $\sharp$ of m tori $T_i^{(m)}$ with a minimal foliation and $c = g - m$ tori $T_j^{(c)}$ with a compact foliation. Complete H_ω to a maximal isotropic subgroup $H \subseteq \ker[\omega]$ such that $\operatorname{rk} H = h$. Namely, denote $h^{(m)} = h - c$; obviously, $0 \leq h^{(m)} \leq m$.

(i) Let $h^{(m)} = 0$. Then just choose all incommensurable periods in all $T_i^{(m)}$.

(ii) Let $h^{(m)} = 1$. Choose the periods $(1, \sqrt{2})$ in $T_1^{(m)}$ and $(1, \sqrt{3})$ in $T_2^{(m)}$. Then $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{21} \rangle$, where z_{i1}, z_{i2} are the basic cycles of $T_i^{(m)}$ corresponding to these periods.

(iii) Let $h^{(m)} = 2$. Similarly, choose the periods $(1, \sqrt{2})$ and $(\sqrt{2}, 1)$ in the first two $T_i^{(m)}$. Then $\ker[\omega|_{\sharp T_i^{(m)}}] = \langle z_{11} - z_{22}, z_{12} - z_{21} \rangle$ is isotropic.

(iv) Let $h^{(m)} = 3$. Choose the periods $(1, \sqrt{2})$, $(\sqrt{2}, -1)$, and $(\sqrt{2}-1, 2\sqrt{2})$ in the first three $T_i^{(m)}$. By Lemma 8, the isotropic subgroup $\langle z_{11} - z_{21} + z_{31}, z_{12} + z_{22} - z_{31}, z_{12} + z_{21} - z_{32}\rangle$ of $\ker[\omega|_{\sharp T_i^{(m)}}]$ is maximal.

(v) Let $h^{(m)} = 2n$, $n \in \mathbb{N}$. Consider n pairs of tori with periods $(\alpha_i, \alpha_i\sqrt{2})$ and $(\alpha_i\sqrt{2}, \alpha_i)$, so that each pair behaves as in (iii) above, but different pairs are incommensurable.

(vi) Let $h^{(m)} = 2n+1$. Choose $n-1$ pairs as in (v) and a triple as in (iv).

By construction, we obtain $h(\ker[\omega]) = c + h^{(m)} = h$.

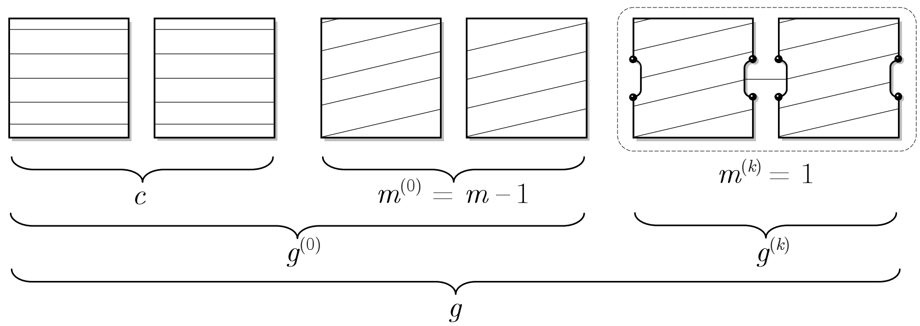

Figure 7. Construction of the foliation in Lemma 14.

Let now $k \geq 2$, thus $g \geq \frac{1}{2}k + 1$. Construct a manifold $M^{(k)}$ with $g^{(k)} = \frac{1}{2}k + 1$, $m(\omega^{(k)}) = 1$, $k(\omega^{(k)}) = k$ as shown in Figure 5 and a manifold $M^{(0)}$ with $g^{(0)} = g - g^{(k)}$, $m(\omega^{(0)}) = m - 1$, $k(\omega^{(0)}) = 0$ as discussed above; see Figure 7. Then $M^{(k)} \sharp M^{(0)}$ has the desired properties. To obtain $h(\ker[\omega]) = h$, $M^{(0)}$ is to be constructed with $h^{(0)} = \min(h, g^{(0)})$ and in $M^{(k)}$, the periods are constructed as in (i)–(vi) above with $h^{(k)} = h - h^{(0)}$ if positive. □

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