

The number of split points of a Morse form and the structure of its foliation

IRINA GELBUKH*

ABSTRACT. Sharp bounds are given that connect split points—conic singularities of a special type—of a Morse form with the global structure of its foliation.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

Consider a smooth closed oriented connected n -dimensional manifold M and a smooth closed differential 1-form ω on it, $d\omega = 0$. By the Poincaré lemma, it is locally the differential of a function: $\omega = df$. We also assume f to be a Morse function; then ω is called a *Morse form*.

Morse functions are smooth functions with non-degenerate singularities. Their set is open and dense in the space of smooth functions [11], i.e., they are “typical” smooth functions. Likewise, Morse forms are “typical” closed 1-forms: their set is open and dense in the space of all closed 1-forms on M .

The set of singularities $\text{Sing } \omega = \{x \in M \mid \omega_x = 0\}$ of a Morse form is finite. On $M \setminus \text{Sing } \omega$ the form ω defines a *foliation* \mathcal{F}_ω constructed as follows: For any $x \in M \setminus \text{Sing } \omega$, the equation $\{\omega_x(\xi) = 0\}$ defines a distribution of the tangent bundle $T_x M$. Since ω is closed, this distribution is integrable; its (connected) integral surfaces are leaves of \mathcal{F}_ω . A leaf $\gamma \in \mathcal{F}_\omega$ *adjoins* a singularity $s \in \text{Sing } \omega$ if $\gamma \cup s$ is connected.

If s has no adjoining leaves (the leaves surrounding it are spheres) then it is called a *center*; we denote the set of all centers by $\Omega_0(\omega)$. If there is exactly one leaf adjoining s then we call s a *transformation point*. If more than one leaf adjoins s (up to four if $\dim M = 2$ and two otherwise) then we call s a *split point*; we denote the set of all split points by $\Omega_1^{sp}(\omega)$.

The motivation behind the terms is that when passing a split point, a leaf splits into two, as in Figure 1 imagining the leaf moving upward; see also Figure 3 (a). In contrast, when passing a transformation point, the leaf keeps its integrity but transforms its shape, as in Figure 3 (b). Our notion of split points coincides with what Levitt [16] referred to as *blocking singularities* because they

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are obstacles for continuation of the local holonomy map. However, we believe that the term “split point” better reflects their simple geometrical meaning: they split one leaf into two.

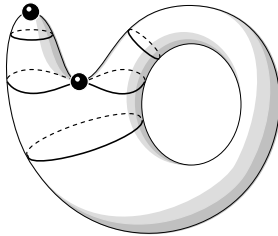


FIGURE 1. A center and a split point.

If $\dim M \geq 3$ then in any non-zero cohomology class there exists a form with only transformation points [15, 20]. Transformation points were thoroughly studied in [1, 15, 16]. We show, however, that it is split points that define the global foliation structure.

Specifically, in this paper we shall study the value

$$d(\omega) = \frac{|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|}{2} + 1,$$

which we show to be non-negative. Generally there are almost no other restrictions on this value: in any suitably defined class it takes all integer and half-integer values greater than the minimum for the class (Proposition 5.1). However, we give lower and upper bound on $d(\omega)$ for some important classes of forms and connect this value with the global structure of the foliation.

The intuition behind the value $|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|$ is that one can locally add any number of center-and-split-point pairs to a foliation without changing its important properties; see Figure 1. Though not every center is attached to the foliation by a split point—see Figure 3 (b), but cf. [2, 3, 20]—we show that the value $d(\omega)$ is still meaningful.

On a 2-dimensional genus g surface M_g^2 , all non-center singularities are split points; the Euler characteristic gives $d(\omega) = g$. On the contrary, if $\dim M \geq 3$ then on a given manifold there exist forms with different values $d(\omega)$.

Our main result demonstrates that while $d(\omega)$ is connected with the properties of a finite number of leaves, it defines the global structure of the foliation: the number $c(\omega)$ of homologically independent compact leaves and the number $m(\omega)$ of minimal components of the foliation. These important characteristics of the foliation have been studied in [1, 14, 17]; various bounds on $c(\omega) + m(\omega)$ have been given in [6, 7, 18]. We show (Theorem 4.1) that

$$c(\omega) + m(\omega) \leq d(\omega); \tag{1.1}$$

what is more, for a “typical” Morse form (from a set open and dense in each cohomology class) the inequality turns into equality:

$$c(\omega) + m(\omega) = d(\omega).$$

For a foliation without minimal components, (1.1) implies an exact bound

$$\mathrm{rk} \omega \leq d(\omega), \tag{1.2}$$

where $\mathrm{rk} \omega$ is the number of independent (over \mathbb{Q}) periods of ω ; all integer and half-integer values greater than this bound are reached on M (Proposition 5.4). This can be rephrased as a condition for existence of minimal components: If $|\Omega_1| - |\Omega_0| < 2 \mathrm{rk} \omega - 2$, then the foliation has a minimal component (Corollary 5.5).

Since $\Omega_1^{sp}(\omega) \subseteq \Omega_1(\omega)$ (the set of all conic singularities), our results imply lower bounds on $|\Omega_1(\omega)| - |\Omega_0(\omega)|$, which is studied in the Novikov theory of closed 1-forms and their singularities [4, 21].

A Morse form is called *generic* if any its leaf adjoins at most one singularity; such forms are “typical” Morse forms: in each cohomology class their set is open and dense [4]. For generic forms, $d(\omega)$ is integer (Proposition 5.6). While on a given manifold M , a generic form can have an arbitrary large number of conic transformation points (Remark 5.2), the number of its split points (up to $|\Omega_0(\omega)|$) is bounded (Proposition 5.6):

$$0 \leq d(\omega) \leq b'_1(M), \tag{1.3}$$

where $b'_1(M)$ is the non-commutative Betti number—the maximal rank of a free factor group of $\pi_1 M$ [16]. All intermediate values are reached on M ; in particular, the bounds are exact.

If a generic form has no minimal components, (1.2) and (1.3) combine into

$$\mathrm{rk} \omega \leq d(\omega) \leq b'_1(M),$$

which is also exact with all intermediate values reached (Corollary 5.7).

The paper is organized as follows. In Section 2 we give necessary definitions and prove some auxiliary lemmas. In Section 3 we introduce the notion of split points and describe some their properties. In Section 4 we prove our main result, showing that the number of split points defines the topology of the foliation. Finally, in Section 5 we show that generally there are almost no restrictions on $d(\omega)$, and give exact inequalities for some important special classes of forms in terms of $\mathrm{rk} \omega$ and $b'_1(M)$.

2. MORSE FORM FOLIATION

Let us introduce, for future reference, some useful notions and facts about Morse forms and their foliations.

2.1. Singularities. A closed 1-form on M is called a *Morse form* if it is locally the differential of a Morse function. Let ω be a Morse form and $\text{Sing } \omega = \{s \in M \mid \omega(s) = 0\}$ the set of its singularities; this set is finite since the singularities are isolated and M is compact.

Since in a neighborhood of a singularity s we have $\omega = df$, the foliation is defined by a Morse function f ; by the Morse lemma there are local coordinates x_1, \dots, x_n such that $x_i(s) = 0$ and $f(x) = f(0) - x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_n^2$. The number k is called the *index* of the singularity s . In a neighborhood of a singularity of index k and $n - k$ the foliation defined by the levels of f has the same topological structure; we denote the set of such singularities by $\Omega_k(\omega)$, $k \leq \frac{n}{2}$.

Singularities $s \in \Omega_0(\omega)$ are called *centers*; a neighborhood of a center foliates into concentric spheres. If ω is exact then $\Omega_0(\omega) \neq \emptyset$ [19]; otherwise in each cohomology class there exists a Morse form without centers: $\Omega_0(\omega) = \emptyset$ [20, Theorem 8.1].

Singularities $s \in \Omega_1(\omega)$ are called *conic*. In a neighborhood of a conic singularity the singular level γ of the corresponding Morse function is (locally) a cone with $\gamma \setminus s$ being not connected. Non-singular levels near s are one-sheeted and two-sheeted hyperboloids; see Figure 5.

At $s \in \Omega_k(\omega)$, $k \geq 2$, the set $\gamma \setminus s$ is connected; nearby non-singular levels are one-sheeted hyperboloids.

2.2. Foliation. On $M \setminus \text{Sing } \omega$ the form ω defines a foliation \mathcal{F}_ω . On the whole M we can define a *singular foliation* (which coincides with \mathcal{F}_ω on $M \setminus \text{Sing } \omega$) as a decomposition of M into leaves; two points $p, q \in M$ belong to the same leaf if there exists a path $\alpha : [0, 1] \rightarrow M$ with $\alpha(0) = p$, $\alpha(1) = q$ and $\omega(\dot{\alpha}(t)) = 0$ for all t . A *singular leaf* contains a singularity. A leaf $\gamma \in \mathcal{F}_\omega$ *adjoins* a singularity s if $\gamma \cup s$ is connected, i.e., if $s \in \overline{\gamma}$ and they belong to the same singular leaf.

A Morse form is called *generic* if each $\gamma \in \mathcal{F}_\omega$ adjoins at most one singularity, i.e., each singular leaf contains a unique singularity. A “typical” Morse form is generic: in each cohomology class on a given M the set of generic forms is open and dense [4].

A leaf $\gamma \in \mathcal{F}_\omega$ is called *compactifiable* if $\gamma \cup \text{Sing } \omega$ is compact; otherwise it is called *non-compactifiable*. If a foliation contains only compactifiable leaves it is called *compactifiable*.

Note that compact leaves are compactifiable. There exists an open neighborhood of a compact leaf γ consisting solely of compact leaves: indeed, integrating ω gives f with $df = \omega$ near γ . Hence, the set covered by all compact leaves is open.

The number of non-compact compactifiable leaves γ_k^0 is finite.

Lemma 2.1 ([10]). *Let $\gamma^0 \in \mathcal{F}_\omega$ be non-compact compactifiable leaf and $\gamma^0 \cup s$ be compact for some $s \in \text{Sing } \omega$. Then there exists a compact leaf $\gamma \in \mathcal{F}_\omega$ which is close to γ^0 .*

A maximal component \mathcal{C}_i^{max} of the foliation is a connected component of the union of all compact leaves. Unless $\text{Sing } \omega = \emptyset$, each maximal component is a cylinder over a compact leaf:

$$\mathcal{C}_i^{max} \cong \gamma_i \times (0, 1),$$

where the diffeomorphism maps γ_i to leaves of \mathcal{F}_ω . The number of maximal components is finite and can be estimated in terms of homological characteristics of M and the number of singularities of ω [7].

A minimal component \mathcal{C}_j^{min} is a connected component of the the set covered by all non-compactifiable leaves. This set is open; it has a finite number $m(\omega)$ of connected components; each non-compactifiable leaf is dense in its minimal component [1]. We say that a minimal component \mathcal{C} contains a singularity $s \in \text{Sing } \omega$ if its punctured neighborhood¹ $U'(s) \subseteq \mathcal{C}$.

Components \mathcal{C}^{max} and \mathcal{C}^{min} are open; their boundaries lie in the union $\bigcup \gamma_k^0 \cup \text{Sing } \omega$ of compactifiable leaves and singularities.

The mentioned sets are mutually disjoint and form a partition of M [5]:

$$M = \left(\bigcup \mathcal{C}_i^{max} \right) \cup \left(\bigcup \mathcal{C}_j^{min} \right) \cup \left(\bigcup \gamma_k^0 \right) \cup \text{Sing } \omega. \quad (2.1)$$

We call *compact singular quasi-leaf* a connected component Υ of the union of non-compact compactifiable leaves and singularities, i.e., of the set $M \setminus (\bigcup \mathcal{C}^{max} \cup \bigcup \mathcal{C}^{min})$; so $\Upsilon = \bigcup \gamma_i^0 \cup \bigcup s_j$, $s_j \in \text{Sing } \omega$. It can be a compact singular leaf or a part of non-compactifiable singular leaf.

2.3. Foliation graph. The configuration formed by maximal components in the decomposition (2.1) is described by the *foliation graph*. Rewrite (2.1) as

$$M = \left(\bigcup \mathcal{C}_i^{max} \right) \cup \left(\bigcup P_j \right),$$

where P_j is a connected component of the union P of all non-compact leaves and singularities.

Since $\partial \mathcal{C}_i^{max} \subseteq P$ consists of one or two connected components, each \mathcal{C}_i^{max} adjoins one or two of P_j . This allows representing M as a connected graph Γ (loops and multiple edges are allowed) whose edges are \mathcal{C}_i^{max} and vertices P_j ; an edge \mathcal{C}_i^{max} is incident to a vertex P_j if $\partial \mathcal{C}_i^{max} \cap P_j \neq \emptyset$; see Figure 2.

We distinguish between two types of vertices: I-vertices, which do not contain minimal components (they consists solely of compactifiable leaves and singularities) and II-vertices, which in addition contain minimal components.

¹A punctured neighborhood $U'(s) = U(s) \setminus s$, where $U(s)$ is a neighborhood.

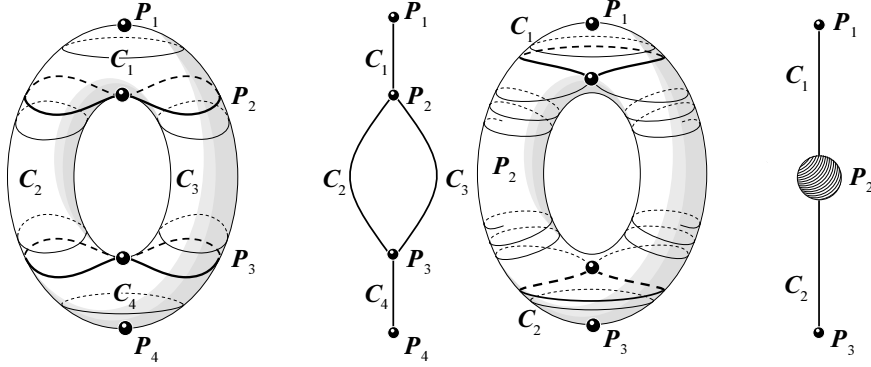


FIGURE 2. Decompositions of the manifold and the corresponding foliation graphs. In the graph on the right, P_2 is a II-vertex; all the other vertices are I-vertices.

The *degree* of a vertex P in the graph is the number of edges incident to this vertex. Geometrically, $\deg P$ is the number of maximal components glued to P . If P is a I-vertex, then it is a compact singular leaf unless $\deg P = 1$, in which case P is a center singularity.

If ω is generic, the vertices of the foliation graph have a rather simple structure:

Lemma 2.2. *Let ω be generic. Then*

- (i) *each I-vertex has degree no greater than 3;*
- (ii) *each II-vertex contains a unique minimal component.*

Proof. (i) In a small neighborhood of a compact singular leaf P the form is exact, so the leaves of the foliation are levels of a Morse function. Since P contains a unique singularity, close levels can have one or two connected components, which are leaves. So $\deg P \leq 3$.

(ii) Consider a connected component ∂ of $\partial\mathcal{C}^{min}$, which is a compactifiable leaf compactified by one singularity. By Lemma 2.1, there exists a compact leaf close to $\partial\mathcal{C}^{min}$. Thus what is attached to \mathcal{C}^{min} by ∂ is an edge. \square

2.4. Graph-theoretic facts. Let Γ be a connected graph with V vertices P_i and E edges. The following simple facts can be found, e.g., in [12].

The degree sum formula states that

$$2E = \sum \deg P_i. \quad (2.2)$$

The *cycle rank* $m(\Gamma)$ of the graph is the number of its independent cycles;

$$m(\Gamma) = E - V + k, \quad (2.3)$$

where k is the number of connected components of Γ . In particular, $E \geq V - k$.

If the graph Γ is considered a 1-dimensional simplicial complex, then

$$m(\Gamma) = b_1(\Gamma), \quad (2.4)$$

the first Betti number.

3. SPLIT POINTS

We call a non-center singularity a *split point* if more than one leaf adjoins it; otherwise it is a *transformation point*. We denote the set of split points by $\Omega_1^{sp}(\omega)$. Obviously, only conic singularities can be split points, $\Omega_1^{sp}(\omega) \subseteq \Omega_1(\omega)$, because for a singular leaf γ at any other singularity s the set $\gamma \setminus s$ is connected.

At a split point, the two parts of the cone (without the singularity) globally lie in different leaves. When passing such a singularity, one leaf splits up into two; see Figure 3 (a). At a conic transformation point, the two parts of the cone happen to globally lie in the same leaf, so that when passing such a singularity the leaf only changes its homotopy type; see Figure 3 (b).

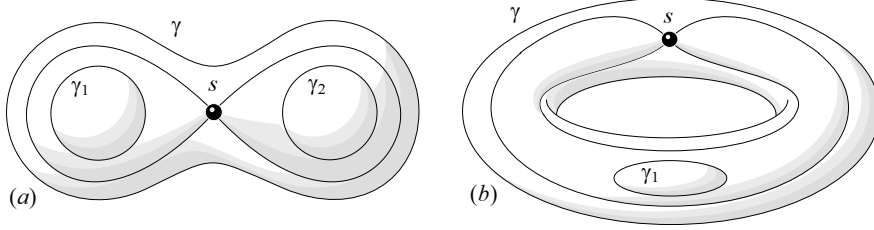


FIGURE 3. (a) s is a split point: the leaf γ splits on it up into γ_1 and γ_2 ; (b) s is a transformation point: the leaf γ transforms on it into γ_1 .

The number of split points defines the structure of the foliation graph. If $|\Omega_1^{sp}(\omega)| = 0$, then the foliation graph is either (a) a chain or circle without II-vertices (\mathcal{F}_ω is compactifiable) or (b) a unique II-vertex (\mathcal{F}_ω is minimal).

The following two statements are useful for the proof of our main theorem. Recall that a compact singular quasi-leaf Υ is a connected component of $M \setminus (\bigcup \mathcal{C}^{max} \cup \bigcup \mathcal{C}^{min})$; $\Upsilon = \bigcup \gamma_i^0 \cup \bigcup s_j$, $s_j \in \text{Sing } \omega$. Denote by $S^\Upsilon \subseteq \Omega_1^{sp}(\omega) \cap \Upsilon$ the set of split points that adjoin only leaves in Υ ; this excludes from $\text{Sing } \omega \cap \Upsilon$ all transformation points and split points adjoining a non-compactifiable leaf.

Definition 3.1 ([22]). A *regular neighborhood* U of $X \subset M$ in M is a locally flat, compact submanifold of M , which is a topological neighborhood of X such that the inclusion $X \hookrightarrow U$ is a simple homotopy equivalence, and X is a strong deformation retract of U .

Since a quasi-leaf is a subcomplex of M , it has a regular neighborhood [13].

Lemma 3.2. *Let $\dim M \geq 3$ and Υ be a compact singular quasi-leaf. Denote by $d(\Upsilon)$ the number of connected components of $U \setminus \Upsilon$, where U is a regular neighborhood of Υ . Then*

$$|S^\Upsilon| \geq d(\Upsilon) - 2. \quad (3.1)$$

Proof. Denote by U_i connected components of $U \setminus \Upsilon$; then $d(\Upsilon) = |\{U_i\}|$; see Figure 4 (

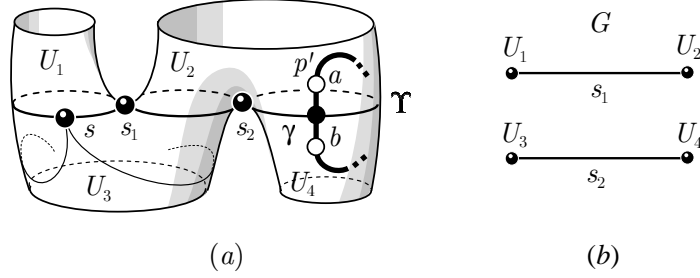


FIGURE 4. (a) The regular neighborhood U of a compact singular quasi-leaf Υ ; $S^\Upsilon = \{s_1, s_2\}$; $s \notin S^\Upsilon$ attaches a minimal component to Υ . A closed path $p' \subset U$ with $[p'] \cdot \Upsilon \neq 0$ is impossible. (b) Graph G is not connected.

We can assume that near $s \in S^\Upsilon$ the boundary ∂U forms a one-sheeted and a two-sheeted hyperboloids, see Figure 5. Consider a graph² $G = \{\{U_i\}, S^\Upsilon\}$, where two vertices U_i, U_j are connected by a conic singularity $s \in S^\Upsilon$ if locally they correspond to the opposite sheets of the two-sheeted hyperboloid; see Figure 4 (b). We will show that G is not connected; then (3.1) follows from (2.3).

Consider an equivalence relation R on $U \setminus \Upsilon$: two points a, b are equivalent if they are connected by a path $p \subset U$ such that $p(t) \in U \setminus \Upsilon$ far from S^Υ , and p is allowed to cross Υ near $s \in S^\Upsilon$ as shown in Figure 5 to connect sheets of the two-sheeted hyperboloid.

Since U is a submanifold of M and U_i is open in U , each U_i is path-connected and thus all points in U_i are equivalent under R . Thus R induces an equivalence relation on the graph G ; its equivalence classes are exactly connected components of G . It remains to show that R has more than one equivalence class.

Consider two close points $a, b \in U$ lying at the opposite sides of a leaf $\gamma \subseteq \Upsilon$; see Figure 4. Suppose they are equivalent under R , i.e., are connected by a path p . For the closed curve $p' = p \cup [a, b]$, the intersection index $[p'] \cdot \Upsilon$ is odd and thus nonzero, which contradicts the fact that Υ is a strong deformation retract of

²Loops and multiple edges are allowed.

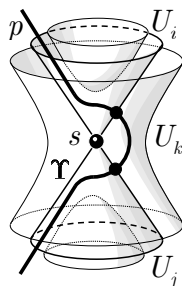


FIGURE 5. U and p' near $s \in S^\Upsilon$.

U . Thus a and b are not R -equivalent, the corresponding U_i belong to different connected components of G , and (2.3) gives (3.1). \square

Proposition 3.3. *Let $\dim M \geq 3$. If a vertex P of the foliation graph Γ contains m minimal components, then*

$$|P \cap \Omega_1^{sp}(\omega)| \geq \deg P + 2m - 2. \quad (3.2)$$

Proof. Consider a graph Γ' whose vertices are (minimal or maximal) components of \mathcal{F}_ω and compact singular quasi-leaves Υ_i , and edges are connected components U_{ij} of $U_i \setminus \Upsilon_i$ for a small regular neighborhood U_i of Υ_i . This is a bipartite graph: an edge can only connect a quasi-leaf Υ with a component \mathcal{C} , but not two Υ or two \mathcal{C} . The value $d(\Upsilon_i)$ from Lemma 3.2 is the degree of the vertex Υ_i in this graph.

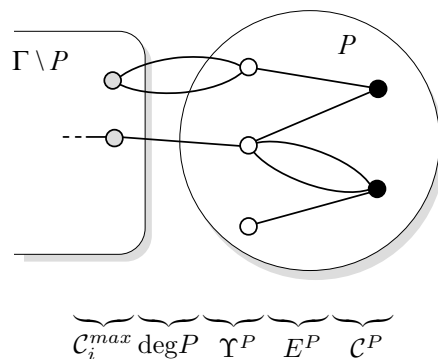


FIGURE 6. A vertex P of the foliation graph Γ as graph Γ' .

A vertex P of the foliation graph Γ is a maximal connected subgraph of Γ' that does not contain any maximal components \mathcal{C}_i^{max} ; see Figure 6. Denote

by $\Upsilon^P \subseteq \{\Upsilon_i\}$, $\mathcal{C}^P \subseteq \{\mathcal{C}_i^{min}\}$, and $E^P \subseteq \{U_{ij}\}$ the sets of vertices and edges belonging to this subgraph; $|\mathcal{C}^P| = m$. Obviously, $\sum_{\Upsilon \in \Upsilon^P} d(\Upsilon) = \deg P + |E^P|$, thus by Lemma 3.2,

$$\left| \bigcup_{\Upsilon \in \Upsilon^P} S^\Upsilon \right| \geq \deg P + |E^P| - 2|\Upsilon^P|. \quad (3.3)$$

Each $U_{ij} \in E^P$ attaches to its Υ a minimal component, thus adding to it at least one split point not from S^Υ , so

$$|\Omega_1^{sp}(\omega) \cap P \setminus \bigcup_{\Upsilon \in \Upsilon^P} S^\Upsilon| \geq |E^P|. \quad (3.4)$$

Adding (3.3) to (3.4), we obtain $|\Omega_1^{sp}(\omega) \cap P| \geq \deg P + 2|E^P| - 2|\Upsilon^P|$, where (2.3) applied to the subgraph P gives $|E^P| \geq |\Upsilon^P| + m - 1$. \square

4. MAIN THEOREM

In the sequel we shall study the properties of the value

$$d(\omega) = \frac{|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|}{2} + 1. \quad (4.1)$$

For a two-dimensional genus g surface M_g^2 , the Euler characteristic gives $d(\omega) = g$.

Recall that a form is generic if each of its singular leaves contains a unique singularity. A minimal component is called *weakly complete* if it contains³ no split points [16]. For $\dim M \geq 3$ the set of generic forms with weakly complete minimal components is known to be open and dense in a cohomology class [8], so such forms are “typical” in their class.

Theorem 4.1. *Let M be a smooth closed oriented manifold and ω a Morse form on it. Then*

$$c(\omega) + m(\omega) \leq d(\omega), \quad (4.2)$$

where $c(\omega)$ is the number of homologically independent compact leaves of the foliation \mathcal{F}_ω and $m(\omega)$ is the number of its minimal components.

For generic forms with weakly complete minimal components it holds

$$c(\omega) + m(\omega) = d(\omega). \quad (4.3)$$

Note that at least for $\dim M \geq 3$ typically the equality holds; in particular, the equation holds for generic forms with compactifiable foliation.

Proof. For $M = M_g^2$ it holds $d(\omega) = g$, so (4.2) follows from $c(\omega) + m(\omega) \leq g$ [7] and (4.3) from $c(\omega) + m(\omega) = g$ for the corresponding class of forms [9]. Assume $\dim M \geq 3$.

³In the sense of Section 2.2: $U'(s) \subseteq \mathcal{C}$.

(i) Let the foliation graph Γ have V vertices P_i and E edges. Then by (2.3) and (2.2) we have $2m(\Gamma) = \sum_i \deg P_i - 2V + 2 = \sum_i (\deg P_i - 2) + 2$. For vertices P that consist of center singularities $s \in \Omega_0(\omega)$ it holds $\deg P = 1$, and for all other vertices Proposition 3.3 gives $\deg P_i - 2 \leq |P_i \cap \Omega_1^{sp}(\omega)| - 2m_i$, where m_i is the number of minimal components in the vertex P_i . We obtain $2m(\Gamma) \leq \sum_i |P_i \cap \Omega_1^{sp}(\omega)| - |\Omega_0(\omega)| - 2m(\omega) + 2$, which together with $m(\Gamma) = c(\omega)$ [7] gives (4.2).

(ii) If ω is generic and its minimal components are weakly complete, then except for $\Omega_0(\omega)$ the inequality in Proposition 3.3 turns into equality, and so do the above inequalities.

Indeed, for a non-center I-vertex P , which is a compact singular leaf, Lemma 2.2 gives $\deg P = 3$ if it contains a (unique) split point and $\deg P = 2$ if it contains a transformation point; this turns (3.2) into equality.

A II-vertex P contains a minimal component \mathcal{C}^{min} . Since minimal components of ω are weakly complete, \mathcal{C}^{min} does not contain split points, and since ω is generic, each connected component ∂_i of $\partial\mathcal{C}^{min}$ contains a unique singularity. By Lemma 2.2, this singularity must be a split point, \mathcal{C}^{min} is the only minimal component in P , and $\deg P = |\{\partial_i\}| = |P \cap \Omega_1^{sp}(\omega)|$, which again turns (3.2) into equality. \square

The theorem allows us to describe the foliation structure in terms of the number of split points. For example, if $|\Omega_1^{sp}(\omega)| < |\Omega_0(\omega)|$ then $c(\omega) = m(\omega) = 0$ and the foliation is compactifiable with all compact leaves being homologically trivial.

If $|\Omega_0(\omega)| \leq |\Omega_1^{sp}(\omega)| \leq |\Omega_0(\omega)| + 1$ and the cohomology class $[\omega] \neq 0$, then Theorem 4.1 and Corollary 5.3 give $c(\omega) + m(\omega) = 1$ and the foliation has a very simple structure; depending on $\text{rk } \omega$, we have:

- (i) If $\text{rk } \omega = 1$ then \mathcal{F}_ω is compactifiable with $c(\omega) = 1$, i.e., all leaves are homologically equivalent, though they do not have to be diffeomorphic; \mathcal{F}_ω is similar to $S^1 \times \text{something}$.
- (ii) If $\text{rk } \omega > 1$ then \mathcal{F}_ω has a unique minimal components and all its compact leaves are homologically trivial; \mathcal{F}_ω is similar to a minimal foliation.

Furthermore, for $\dim M \geq 3$ foliations with such a simple structure are known to exist in any cohomology class $[\omega]$, namely:

- $\text{rk}[\omega] = 1$: There exists a compactifiable foliation with $c(\omega) = 1$.
- $\text{rk}[\omega] > 1$: There exists a minimal and uniquely ergodic foliation [1].

Indeed, in a non-zero cohomology class there exists a Morse form without centers [20]. Among such forms there exists a form with $\Omega_1^{sp}(\omega) = \emptyset$ [1]; as above, Theorem 4.1 and Corollary 5.3 give $c(\omega) + m(\omega) = 1$. The fact for

$\text{rk } \omega = 1$ follows from (i) above. If $\text{rk } \omega > 1$, then by (ii) above the cycle rank of the foliation graph $m(\Gamma) = c(\omega) = 0$, i.e., Γ is a tree with exactly one II-vertex P that contains the minimal component. If \mathcal{F}_ω had any compact leaves, then Γ would have edges and thus terminal vertices other than P , which would be centers in \mathcal{F}_ω . Thus the foliation is minimal; by [1], it is uniquely ergodic.

5. BOUNDS ON $d(\omega)$

Since for M_g^2 it holds $d(\omega) = g$, in the sequel we shall assume $\dim M \geq 3$ unless otherwise stated. We shall show that in the general case there are no restrictions on $d(\omega)$ besides a very non-restrictive lower bound. However, compactifiable foliations allow a stronger lower bound on $d(\omega)$ and generic forms allow an upper bound. Naturally, generic compactifiable foliations allow both.

5.1. No upper bound. Levitt [15] proved that a small local perturbation within the cohomology class can turn all split points into transformation points. Figure 7 shows the converse: a local—though not small—perturbation within the cohomology class can turn all conic transformation points into split points and centers; each destroyed conic transformation point adds $\frac{1}{2}$ to $d(\omega)$.

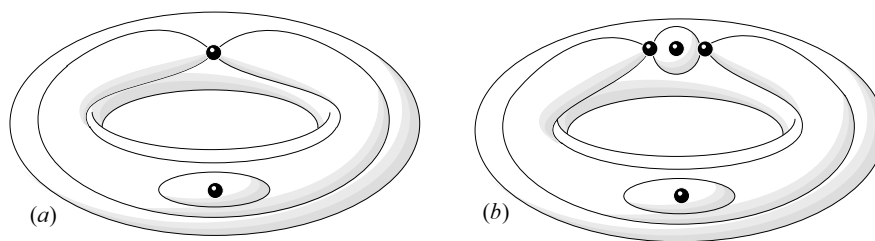


FIGURE 7. A transformation point can be turned into two split points and one center. Gluing (a) and (b) together by the boundaries into an $S^{n-1} \times S^1$ gives $d(\omega) = \frac{1}{2}$.

In this way an unlimited number of conic transformation points can be added to any foliation:

Proposition 5.1. *In a class of forms with given $[\omega]$, $c(\omega)$, $m(\omega)$, and compactifiability, $d(\omega)$ takes all integer and half-integer values greater than the minimum for this class.*

Proof. Any foliation \mathcal{F}_ω can be locally modified preserving all its important characteristics so that $d(\omega')$ be arbitrary large and take all integer and half-integer values $d \geq d(\omega)$.

Indeed, consider a singular leaf shown in Figure 8. Its inside is $S^1 \times D^{n-2}$; let it be foliated as shown in Figure 7(b). Its outside leaves are spheres. Any

number of such solid spheres can be attached through split points to the foliation as shown in Figure 1. Each such sphere adds to $\text{Sing } \omega$ one transformation point of index 2 (which happens to be conic for $\dim M = 3$; see Figure 8), two centers, and three split points (one of them attaches the sphere to the original foliation), which increases $d(\omega)$ by $\frac{1}{2}$.

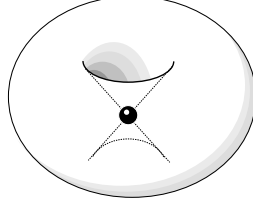


FIGURE 8. A sphere transforms into $S^1 \times S^{n-2}$.

This operation preserves the cohomology class (thus $\text{rk } \omega$), compactifiability, $c(\omega)$, and $m(\omega)$ (as well as some other properties of \mathcal{F}_ω). \square

Note that the form constructed in Proposition 5.1 is not generic.

Remark 5.2. In a class of forms with given $d(\omega)$, $[\omega]$, $c(\omega)$, $m(\omega)$, compactifiability, and genericity, the number $|\Omega_1^{tr}(\omega)|$ of conic transformation points takes all values greater than the minimum for this class.

The proof is as in Proposition 5.1, with Figure 7(a) used instead of Figure 7(b).

5.2. Lower bound. Recall that $[\omega]$ is the cohomology class of ω ; $[\omega] = 0$ means globally $\omega = df$.

Corollary 5.3. *It holds*

$$d(\omega) \geq \begin{cases} 0 & \text{if } [\omega] = 0, \\ 1 & \text{otherwise.} \end{cases}$$

If $\dim M \geq 3$ then in each cohomology class the bound is exact and all greater integer and half-integer values are reached.

Proof. While it is not obvious from the definition (4.1), Theorem 4.1 implies $d(\omega) \geq 0$.

Suppose $d(\omega) \leq \frac{1}{2}$. Theorem 4.1 implies $c(\omega) = m(\omega) = 0$, thus \mathcal{F}_ω is compactifiable with all compact leaves being homologically trivial. Then $[\omega] = 0$, because for a compactifiable foliation it holds [7]

$$\text{rk } \omega \leq c(\omega). \tag{5.1}$$

Exactness for the case $[\omega] = 0$ follows from the existence of compactifiable foliation with $c(\omega) = \text{rk } \omega$ [6, Theorem 8]; Theorem 4.1 gives $d(\omega) = 0$. On the other hand, in any non-zero cohomology class there exists a Morse form ω without split points and centers: $\Omega_1^{sp}(\omega) = \Omega_0(\omega) = \emptyset$ [15]; $d(\omega) = 1$.

Finally, all greater values are reached by Proposition 5.1. \square

For $d(\omega) = 0$ or $\frac{1}{2}$, Corollary 5.3 gives $\omega = df$.

5.3. Lower bound for compactifiable foliations. If \mathcal{F}_ω is compactifiable, then $\text{rk } \omega \leq b'_1(M)$, the non-commutative Betti number [15]; indeed, $\text{rk } \omega \leq c(\omega)$ [7] and $c(\omega) \leq b'_1(M)$ [6].

For $M = M_g^2$ it holds $b'_1(M_g^2) = g$ [6], so for a compactifiable foliation on M_g^2 it holds $\text{rk } \omega \leq d(\omega) = g$.

Proposition 5.4. *Let $\dim M \geq 3$ and \mathcal{F}_ω be compactifiable. Then*

$$\text{rk } \omega \leq d(\omega);$$

On a given M this lower bound on $d(\omega)$ is exact and all larger values of $d(\omega)$ are reached. Namely, for any non-negative integer $r \leq b'_1(M)$ and any integer or half-integer $d \geq r$, on M there exists a form ω with $\text{rk } \omega = r$ and $d(\omega) = d$.

Proof. By (5.1) and Theorem 4.1 we have $\text{rk } \omega \leq c(\omega) \leq d(\omega)$, which gives the bound. Let now $r \leq b'_1(M)$. There exists a generic form with compactifiable foliation such that $c(\omega) = r$ [6, Theorem 8]. Furthermore, for the same foliation we can choose ω such that $\text{rk } \omega = c(\omega)$ [7, Theorem 4.1]. On the other hand, (4.3) gives $c(\omega) = d(\omega)$.

Finally, all values of $d(\omega) > r$ are reached by Proposition 5.1. \square

In particular, a form with compactifiable foliation and a large $\text{rk } \omega$ has many split points—many more than centers. While in any cohomology class with $\text{rk } \omega > 1$ there exist forms without split points, their foliations are minimal [15]; thus the only forms without split points with compactifiable foliation are rational forms—those with $\text{rk } \omega = 1$, i.e., for some $k \in \mathbb{R}$, $k[\omega] \in H^1(M, \mathbb{Z})$.

Given (4.1), Proposition 5.4 can be considered as a condition for existence of minimal components:

Corollary 5.5. *If $|\Omega_1| - |\Omega_0| < 2 \text{rk } \omega - 2$, then the foliation has a minimal component: $m(\omega) > 0$.*

5.4. Upper bound for generic forms. Recall that a form is called generic if each its singular leaf contains a unique singularity (such forms are “typical”); $b'_1(M)$ is the first non-commutative Betti number: the maximal rank of a free factor group of $\pi_1 M$ [15].

Proposition 5.6. *Let ω be generic. Then $d(\omega)$ is integer and*

$$0 \leq d(\omega) \leq b'_1(M); \tag{5.2}$$

on a given M the bounds are exact and all integer intermediate values are reached.

Proof. Both the fact that $|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|$ is even and the bounds on $|\Omega_1^{sp}(\omega)| - |\Omega_0(\omega)|$ for generic forms were proved in [16].

It was shown in [6, Theorem 8 and Remark 12] that for any c within the bounds (5.2) there exists a generic form ω with $c(\omega) = c$ and $m(\omega) = 0$; thus ω is trivially π_1 -stable. By Theorem 4.1, $d(\omega) = c(\omega) + m(\omega) = c$. \square

Corollary 5.7. *If ω is generic and \mathcal{F}_ω compactifiable, then*

$$\text{rk } \omega \leq d(\omega) \leq b'_1(M)$$

and for any non-negative integer r , d such that $r \leq d \leq b'_1(M)$, on M there exists a generic form ω with $\text{rk } \omega = r$ and $d(\omega) = d$.

Proof. On M there exist generic forms with any $c(\omega)$ between 0 and $b'_1(M)$ [6], and for the same \mathcal{F}_ω we can choose a form ω' with any $\text{rk } \omega'$ between 0 and $c(\omega)$ [7]. Finally, $d(\omega') = c(\omega')$ by Theorem 4.1. \square

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* CIC, IPN, 07738, DF
MEXICO
E-mail address: gelbukh@member.ams.org