Topology of the Reeb graph

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Abstract

Reeb graph of a function is a space obtained by contracting the connected components of the level sets of the function to points, endowed with the quotient topology (plus an additional structure in the case of a smooth function). This notion is useful in topological classification of functions and, under the name of Lyapunov graph, in theory of dynamical systems. It also finds practical applications in computer graphics, shape analysis, machine learning, big data analysis, and geometric model databases. We give a criterion for the Reeb graph to have the structure of a finite graph (generally it is not: we give counterexamples) and describe general properties of such graphs. We also consider the realization problem: the conditions for a finite graph to be the Reeb graph of a function of some class, such as any smooth function, Morse, Morse-Bott, or round function.

Contents

Introduction

Concept of Reeb graph

- **Reeb graph** was introduced by George Reeb (1946), in that time
  - only for simple Morse functions
  - on closed manifolds

  as a **quotient space**: connected components of level sets contracted to points
- He noted that it is a 1-dim (CW) complex: a **finite graph** (with multiple edges)
  - for the type of functions he considered
  - this is used in all modern applications of the Reeb graph
- Lots of papers on applications, but only recently research on its **topology**

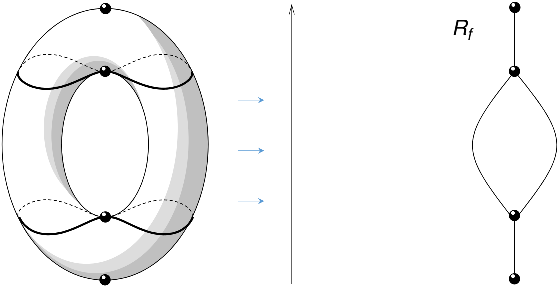
Topic of this presentation: **Reeb graph topology**

Reeb graph of a smooth function

- X is a topological space; $f : X \to \mathbb{R}$ is a continuous function
- **Contour** of f: connected component of its level set $f^{-1}(y)$
- $x \sim y$ is an equivalence relation: $x, y \in$ the same contour of f

Definition

The **Reeb graph** R_f is the quotient space $X/\sim$, endowed with the quotient topology. For smooth functions: image of a critical contour is called a **vertex**.



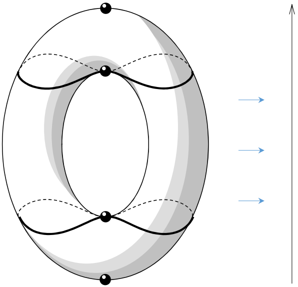
Geometric meaning of the Reeb graph

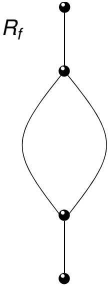
In some “good” cases, Reeb graph is indeed a **graph**

- non-vertices form **edges**
- much more on this, later

This graph shows the **evolution** of the level sets:

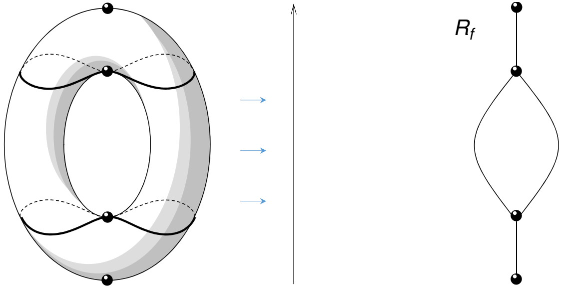
- contours can **split** into two or more
- contours can **merge** into one





Orientation of the Reeb graph

- f defines $f_R : R_f \to \mathbb{R}$ such that $f = f_R \circ \varphi$: $\begin{array}{ccc} X & \xrightarrow{f} & \mathbb{R} \\ \downarrow & \nearrow f_R & \\ R_f & & \end{array}$
- On edges f_R is monotonous; direction of growth defines an **acyclic orientation**
- Usually R_f is simpler than X, and f_R simplifies f.



Applications: computer science

R_f describes behavior of the function: evolution of the topology of the level sets.

- Computer graphics and shape analysis:
  - compact shape descriptor: topological info on level sets of a function on the shape
  - shape $\rightarrow$ appropriate simple Morse function $\rightarrow$ Reeb graph [Biasotti et al. (2008)]
- Geometric model databases
- Data analysis and visualization
- Machine learning and big data analysis:
  - R_f + metric derived from the data $\Rightarrow$ hidden structure in data [Ge et al. (2011)]

Lots of papers and algorithms [Edelsbrunner and Harer (2010)].

Applications: mathematics

Reeb graph also has applications in pure mathematics:

- For **topological classification** of
  - Morse functions [Arnold (2007)]
  - functions with isolated critical points [Sharko (2006), Ukrainian school]
  - simple Morse–Bott functions [Martínez-Alfaro et al. (2016, 2018)]
- In the theory of **dynamical systems**
  - Lyapunov function (*Lyapunov graph*), in the context of gradient-like flows [Bertolim et al. (2004)]
  - Lyapunov graph: dynamical information of a flow, topological info of its phase space [Lima et al. (2019)]

Topology of the Reeb graph

Reeb graph as a finite graph

- M is a **manifold**, $f: M \to \mathbb{R}$ is a **smooth function**
- A finite **graph** can have multiple edges and loops: a 1-dimensional CW complex

Definition

The Reeb graph R_f **has the structure of a finite graph** G, if there is a homeomorphism $h: R_f \to G$ mapping vertices of R_f bijectively to vertices of G.

We will say that R_f is **isomorphic** to G or just R_f **is** G (abuse of language).

Example

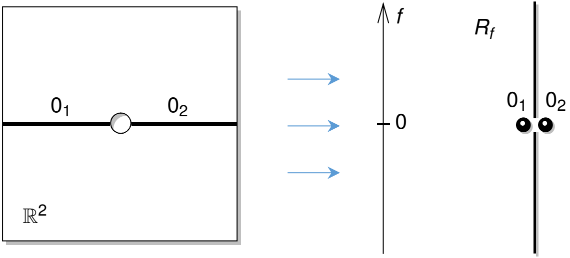
- The Reeb graph of a simple Morse function has the structure of a finite graph [Reeb (1946)]
- The Reeb graph of a simple Morse–Bott function on a surface has the structure of a finite graph [Martínez-Alfaro et al. (2016)]

Counterexample 1: Reeb graph is not a finite graph

Generally, the Reeb graph is not a graph.
This quotient space can be ill-behaved even for very good functions:

Example

Let $M = \mathbb{R}^2 \setminus \{(0,0)\}$ and $f(x,y) = y$ be the projection.
Then R_f is the line with two origins (bug-eyed line).
Not a graph, even non-Hausdorff.



The problem is that the manifold is **not compact**.

Counterexample 2: Reeb graph is not a finite graph

Even on a compact manifold:

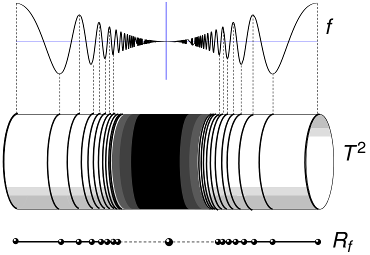
Example

Torus T^2 with coordinates (x, t).

Function $f : S^1 \to \mathbb{R}$, $f(t) = e^{-\frac{1}{t^2}} \cos(\frac{1}{t})$ near 0, glued smoothly near $\pm\pi$.

R_f is S^1, but infinite number of vertices: not a finite graph.

Even not connected in graph topology: central vertex is isolated.



The problem is that the function has an **infinite number of critical values**.

When is the Reeb graph a finite graph?

Theorem (Saeki (2021))

Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function. Then:
R_f has the structure of a finite graph $\Leftrightarrow$ f has a finite number of critical values.

This makes it possible:

- to work with a wide class of functions, including Morse–Bott and round functions;
- to study these functions using graph theory;
- to generalize facts from Morse functions to more general classes of functions.

Properties of the Reeb graph that is a finite graph

If R_f is a **finite graph**, we study how its characteristics are related with:

- the type of the **manifold** M,
- the **class of the function** f,

Realization problem:

- Is any finite graph **isomorphic** to the Reeb graph of some function? **– No**
- Is any finite graph **homeomorphic** to the Reeb graph of some function? **– Yes**

Corank of the fundamental group of a manifold

Definition

The **co-rank** of a finitely generated group G is
the maximum rank of a free quotient group of G, i.e.,
the maximum rank of a free group F such that there exists an epimorphism $\phi : G \twoheadrightarrow F$.

Since $\pi_1(M)$ of a compact manifold M is finitely generated, consider $\operatorname{corank}(\pi_1(M))$:

- Closed orientable surface: **genus**,
- 3-manifold: **cut number**.

See methods of calculation in [Gelbukh (2017)].

Example

- $\operatorname{corank}(\pi_1(S^n)) = 0$, for a sphere, $n \geq 2$,
- $\operatorname{corank}(\pi_1(T^n)) = 1$, for a torus, $n \geq 2$,
- $\operatorname{corank}(\pi_1(M_g)) = g$, for orientable closed surface of genus g,
- $\operatorname{corank}(\pi_1(N_g)) = [\frac{g}{2}]$, for non-orientable closed surface of genus g,

Characteristics of the Reeb graph and the manifold

$b_1(X)$ is the first Betti number, for a graph: cycle rank.

Theorem (Gelbukh (2019))

Let M be a closed manifold,
$f : M \to \mathbb{R}$ a smooth function with finite number of critical values. Then:

$$b_1(R_f) \le \operatorname{corank}(\pi_1(M)).$$

This inequality is tight: on a given M there exists f (simple Morse) with equality.

For simple Morse functions on a surface, proved by Cole-McLaughlin et al. (2004).

Example

- $\operatorname{corank}(\pi_1(S^n)) = 0$, so R_f of a function on a sphere is a tree.
- $\operatorname{corank}(\pi_1(T^n)) = 1$, so the cycle rank of R_f on a torus is 0 or 1: $b_1(R_f) \le 1$.

Realization problem:
When is a finite graph the Reeb graph of a function?

Realization: smooth function

Realization problem: Is any finite graph the Reeb graph of some function?

No. But yes for graphs **without loops** (edge with both endpoints at the same vertex):

Theorem (Masumoto and Saeki (2011))

Let G be a finite graph. Then:
there is a smooth function $f : M \to \mathbb{R}$ *such that* R_f *is G* $\Leftrightarrow$ *G has no loops.*

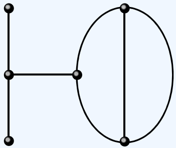
Indeed, R_f that is a finite graph has an acyclic orientation $\Rightarrow$ no loops.

Realization: Morse function. Counterexample

Realization problem in some **class of functions**: additional conditions on the graph.

Example (Sharko (2006))

Not R_f of any Morse function. Not even function with finite $\mathrm{Crit}(f)$:

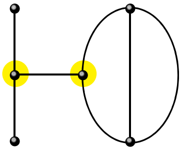


Why? Let's see. First, some graph theory...

Some graph theory

Definition

- **Cut vertex**: $G \setminus v$ has more connected components. Isolated vertex is not.

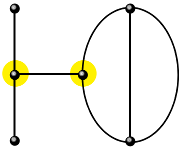


2 cut vertices, 4 blocks, 3 of them leaf blocks

Some graph theory

Definition

- **Cut vertex**: $G \setminus v$ has more connected components. Isolated vertex is not.
- **Biconnected** graph: connected, without cut vertices.



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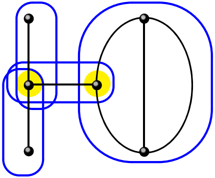
- **Block** of a graph: maximal biconnected subgraph. Isolated vertex is a block.



2 cut vertices, **4 blocks**, 3 of them leaf blocks

Some graph theory

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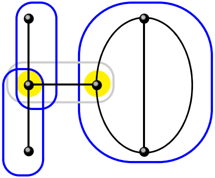
- **Biconnected** graph: connected, without cut vertices.
- **Block** of a graph: maximal biconnected subgraph. Isolated vertex is a block.
- **Leaf block**: a block with at most one cut vertex.

Blocks are attached to each other at shared vertices = cut vertices of the graph. (This forms the **block-cut tree**, of which leaf blocks are leafs—hence the term.)

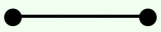


2 cut vertices, 4 blocks, **3 of them leaf blocks**

Realization: Morse function

- Closed manifold
- Morse function
- Generally, function with finite number of critical points

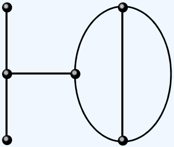
Theorem (Michalak (2018) + Gelbukh (submitted1))

G is R_f of a smooth function with finite $\mathrm{Crit}(f)$ on a closed manifold $\Leftrightarrow$
G is finite, no loops, all leaf blocks are  *(K_2).*

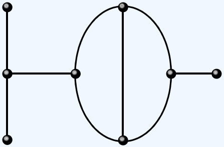
f can be chosen Morse.

To make a given graph realizable by a Morse f, add K_2 to each non-K_2 leaf block:

Example



☹ non-K_2 leaf block



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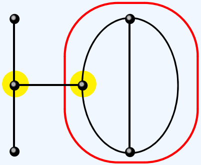

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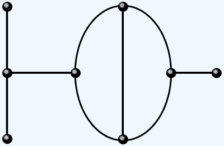

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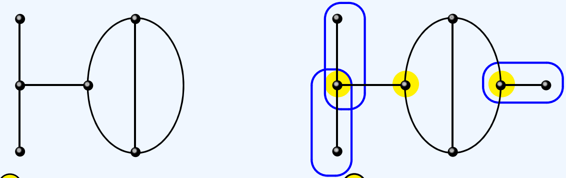


☹ non-K_2 leaf block ☺ all K_2 leaf blocks

Realization: Morse–Bott function

Morse–Bott function; generally, function with finite number of critical **submanifolds**:

Theorem (Gelbukh (submitted2))

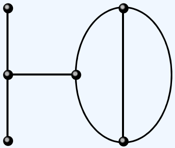
For any given $n \geq 2$,
G is R_f of a smooth f with $\mathrm{Crit}(f) =$ finite no. of submanifolds, on closed n-manifold $\Leftrightarrow$
G is finite, no loops, and

- *each leaf block L has a vertex of degree ≤ 2,*
- *two such vertices if L is a non-trivial (has an edge) connected component of G.*

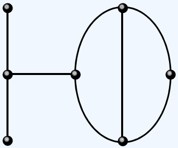
f can be chosen Morse–Bott.

To make G realizable by a Morse–Bott f, subdivide an edge in leaf blocks where missing:

Example



☹ no vertex of degree ≤ 2



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Example

☹ no vertex of degree ≤ 2 ☺ all leaf blocks have ≤ 2

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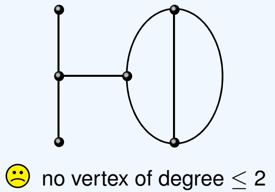
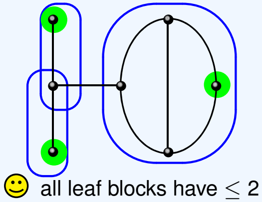




Realization: Morse–Bott function (homeomorphism)

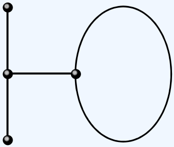
Morse–Bott functions play a special role in the Reeb graph theory:

Theorem (Gelbukh (in press))

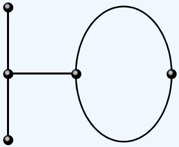
Any *finite graph is* ***homeomorphic*** *to the Reeb graph of a Morse–Bott function.*

True even for a graph with loops: can subdivide a loop by a vertex of degree 2.

Example



☹ loop: no any function



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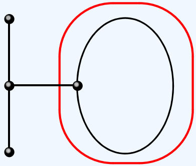

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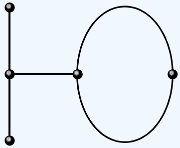

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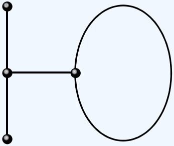

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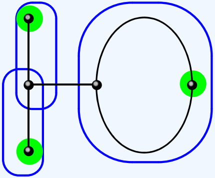


☺ no loop, Morse–Bott function

Realization: round function

Definition

Round function: smooth function $f : M \to \mathbb{R}$ on a closed manifold M, with $\mathrm{Crit}(f) = \bigcup S^1$, a finite number of disjoint **circles**.

This time, the structure of R_f depends on manifold:

- dimension
- whether orientable

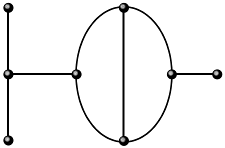
Theorem

G is R_f of a round function on M^n $\Leftrightarrow$ G is finite, no loops, and

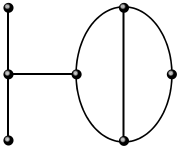
$$\text{each leaf block} \begin{cases} \text{has a non-cut vertex of } \deg v = 2 & \text{if } n = 2 \text{, orientable surface} \\ \text{has a non-cut vertex of } \deg v \leq 2 & \text{if } n = 2 \text{, non-orientable surface} \\ \text{is } \bullet\!\!-\!\!\!-\!\!\!-\!\!\bullet \ (K_2) & \text{if } n \geq 3 \end{cases}$$

Realization: conclusion

A graph can be realized by functions of **different classes** and on **different manifolds**:



f	$n=2$ orient	$n=2$ non-or	$n \geq 3$
Morse	+	+	+
Morse–Bott	·	·	·
round	·	+	+



f	$n=2$ orient	$n=2$ non-or	$n \geq 3$
Morse	·	·	·
Morse–Bott	+	+	+
round	·	+	·

$b_1(G) = 2 \Rightarrow \operatorname{corank}(\pi_1(M)) \geq 2$. E.g., for surface: genus $\geq \begin{cases} 2 & \text{orientable,} \\ 4 & \text{non-orientable.} \end{cases}$

Future work: Generalizations

Generalizations

What's next?

- Smooth functions with an **infinite number of critical values**:

Theorem (Gelbukh (2018))

Let M be closed orientable, f smooth. Then:

$$b_1(R_f) \leq \operatorname{corank}(\pi_1(M)).$$

This inequality is tight: on a given M there exists f (simple Morse) with equality.

- Smooth functions of **other classes**: we saw (simple) Morse, Morse–Bott, round, ...
- Other **continuous functions** on a topological space X
- Functions on manifolds M or spaces X with given properties
- Functions $X \to Y \neq \mathbb{R}$ (case $M \to S^1$ has been studied)

Conclusion

Conclusion

- Reeb graph: connected components of level sets → points, quotient topology
  - Smooth function: some points marked as *vertices*: preimage contains a critical point
- Applications in:
  - Mathematics: topological classification of functions; dynamical systems
  - Computer science: computer graphics, shape analysis, data analysis, machine learning
- In good cases, Reeb graph "is" a finite graph—but not always:
  - For smooth functions, exactly when finite number of critical values
- When it is, its graph-theoretic properties depend on the function and manifold
- Realization problem: for a given graph, is it the Reeb graph of some function?
  - function of a given class: (simple) Morse, Morse-Bott, round
- Interesting area, not well-studied yet in mathematical aspects

Good topic for your own research!

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Thank you! :)

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