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A Test for Compactness of a Foliation

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ABSTRACT. We investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. We introduce the notion of the degree of compactness and prove a test for compactness.

In the present paper we investigate foliations on smooth manifolds that are determined by a closed 1-form with Morse singularities. The problem of investigating the topological structure of level surfaces for such a form was posed by S. P. Novikov in [1]. This problem was treated in [2–5]. The present paper is devoted to the compactness problem for level surfaces. We introduce the notion of degree of compactness and prove a test for compactness expressed in the terms of the degree.

In §1 we give the necessary definitions and define the degree of compactness. The central result of the paper, i.e., the test for compactness of a foliation, is proved in §2. In §3 we present some consequences: a relation between the degree of compactness and the degree of irrationality of the form, and a more detailed investigation of the two-dimensional case.

The present paper is a natural continuation of [6].

§1. Preliminary definitions

Consider a smooth compact n-dimensional manifold M and a closed 1-form ω on M with nondegenerate isolated singularities.

Definition 1 [7]. A point $p \in M$ is said to be a *regular singularity* of the differential form ω if in a neighborhood $O(p)$ we have $\omega = df$, where f is a Morse function with a singularity at p. There exist, therefore, coordinates $x^1, \dots, x^n$ such that in this neighborhood we have

$$\omega = \sum_{i=1}^{k} x_i dx^i - \sum_{i=k+1}^{n} x_i dx^i.$$

The number $\min(k, n-k)$ is called the *index* of the singular point.

On the set $M - \operatorname{Sing}\omega$ the form ω determines a foliation $\mathcal{F}_\omega$ of codimension 1. If the index of the singular point P is equal to zero, then there exists a foliation of a neighborhood of P into spheres. If $\operatorname{ind} P = 1$, then there exists a fiber that becomes locally arcwise connected after adding the singular point P. This fiber is called the *canonical fiber*. For $\operatorname{ind} P > 1$ all the fibers in the neighborhood of P are locally arcwise connected.

The foliation $\mathcal{F}_\omega$ contains fibers of three kinds [7]:

1) compact fibers admitting a neighborhood consisting of diffeomorphic fibers;
2) conic fibers, i. e., fibers that may be made locally arcwise connected in a neighborhood of a singular point by adding this singular point to the fiber;
3) all the other noncompact fibers.

Below we assume that the singular point P belongs to the fiber, and thus all the fibers are arcwise connected.

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