A test for non-compactness of the foliation of a Morse form

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In this paper we study foliations determined by a closed 1-form with Morse singularities on smooth compact manifolds. The problem of the topological structure of the level surfaces of such forms was posed by Novikov in [1], and has been studied in [2]–[5]. In the present article we investigate the problem of the existence of a non-compact leaf, verify a test for non-compactness of a foliation in terms of the degree of irrationality of the form ω, and show that the non-compactness of a foliation is a case of general position.

We consider a compact manifold M and a closed 1-form ω with Morse singularities defined on it. The closed form ω determines a foliation of codimension 1 on the set $M - \operatorname{Sing}\omega$. Correspondingly, a foliation $\mathcal{F}_\omega$ with singularities is obtained on M by adjoining the singular points to $M - \operatorname{Sing}\omega$. We say that a leaf $\gamma \in \mathcal{F}_\omega$ is compact if it is a non-singular compact leaf or can be compactified by adding singular points. The foliation $\mathcal{F}_\omega$ is said to be compact if all its leaves are compact.

Definition 1. Let γ be a non-singular compact leaf of $\mathcal{F}_\omega$, and consider the map $\gamma \to [\gamma] \in H_{n-1}(M)$. Then the image of the set of compact leaves under this map generates a subgroup of $H_{n-1}(M)$. Denote it by H_ω.

The foliation $\mathcal{F}_\omega$ is characterized by the condition that $\gamma \cap \gamma' = \varnothing$ for $\gamma, \gamma' \in \mathcal{F}_\omega$. We consider the group $H_{n-1}(M)$ and the intersection operation for homology classes,

$$H_{n-1}(M) \times H_{n-1}(M) \to H_{n-2}(M).$$

If γ and γ' are non-singular compact leaves of $\mathcal{F}_\omega$, then $[\gamma] \circ [\gamma'] = 0$.

Definition 2. Let $H \subset H_{n-1}(M)$ be a subgroup such that $x \circ y = 0$ for all $x, y \in H$. We say that H is an isotropic subgroup with resepct to the intersection operation for cycles. An isotropic subgroup H is said to be *maximal* if for all $x \notin H$ there is a $y \in H$ such that $x \circ y \neq 0$. The rank of a maximal isotropic subgroup is denoted by $h_0(M)$.

The subgroup H_ω of compact leaves is clearly an isotropic subgroup of $H_{n-1}(M)$. It can be shown that $h_0(M)$ is not uniquely determined. Let $h_0^{\max}(M)$ denote the maximal value of $h_0(M)$.

Definition 3 [1]. The degree of irrationality of the form ω is defined to be

$$\operatorname{dirr}\omega = \operatorname{rk}_{\mathbb{Q}}\left\{\int_{z_1}\omega, \dots, \int_{z_m}\omega\right\} - 1,$$

where $z_1, \dots, z_m$ is a basis in $H_1(M)$.

It was proved in [1] that if $\operatorname{dirr}\omega = 0$, then the foliation $\mathcal{F}_\omega$ is compact. The following test for non-compactness of a foliation on M_g^2 was proved in [6].

Theorem 2 [6]. *If* $\operatorname{dirr}\omega \geq g$ *on* M_g^2, *then the foliation* $\mathcal{F}_\omega$ *has a non-compact leaf.*

We prove a generalization of this theorem to a manifold of arbitrary dimension.

Theorem. *If the foliation* $\mathcal{F}_\omega$ *on a manifold* M *is determined by a Morse form* ω *and* $\operatorname{dirr}\omega \geq h_0^{\max}(M)$, *then* $\mathcal{F}_\omega$ *has a non-compact leaf.*

Proof. Suppose that the foliation $\mathcal{F}_\omega$ is compact, and consider a non-singular leaf $\gamma \in \mathcal{F}_\omega$. Since $\omega|_\gamma = 0$, the form is exact in some neighbourhood of $\gamma : \omega = df$ and the leaves of $\mathcal{F}_\omega$ are the levels of the function f. In a neighbourhood where $\operatorname{grad} f \neq 0$ (that is, the form ω is non-singular) all the leaves are diffeomorphic. Thus, each non-singular leaf γ has a neighbourhood consisting of leaves diffeomorphic to it.

Let $\mathcal{O}(\gamma)$ denote a maximal such neighbourhood. This neighbourhood is the cylinder with generator $\gamma : \mathcal{O}(\gamma) = \gamma \times \mathbb{R}$. We consider its closure $V = \overline{\mathcal{O}(\gamma)}$. The boundary of V contains at least one critical point of the form ω. Let γ' be another non-singular compact leaf of $\mathcal{F}_\omega$. Obviously, the cylinders $\mathcal{O}(\gamma)$ and $\mathcal{O}(\gamma')$ either are disjoint or coincide, and $V \cap V' \subset \partial V \cup \partial V'$.

Since ω is a Morse form and M a compact manifold, there are finitely many singular points. Consequently, the number of different cylinders $\mathcal{O}(\gamma)$ is also finite. Thus, the manifold M on which the compact foliation is given can be represented in the form

$$M = \bigcup_{i=1}^{N} \mathcal{O}(\gamma_i) \bigcup_{k=1}^{K} \gamma_k^0 \bigcup_{j=1}^{I} p_j,$$

where the p_j are the singular points of ω, and the γ_k^0 are the singular leaves of $\mathcal{F}_\omega$.

Let $V_i = \overline{\mathcal{O}(\gamma_i)}$ and let $T = \bigcup_{i=1}^{N} \partial V_i$. Obviously, $p_j \in T$ and $\gamma_k^0 \in T$. Then $M = \bigcup_{i=1}^{N} V_i$, and $V_i \cap V_j \subset T$.

We investigate the connection between the homology $H_1(M)$ and the representation $M = \bigcup_{i=1}^{N} V_i$. Using the exact Mayer–Vietoris sequence, we can show that $H_1(M) = \langle i_{k*} H_1(V_k), D[\gamma_k], k = 1, \dots, N\rangle$, where $i_k : V_k \to M$. Since $V_k = \overline{\mathcal{O}(\gamma_k)}$, $\partial V_k \cap \operatorname{Sing}\omega \neq \varnothing$, and the form ω is locally exact, by considering Morse surgery at a singular point we deduce that $H_1(M) = \langle j_{k_*} H_1(\gamma_k), D[\gamma_k], k = 1, \dots, N\rangle$, where $j_k : \gamma_k \to M$ is an embedding.

Let us compute the periods of the form ω. It suffices to consider a $z \in H_1(\gamma_i)$ with $z = D[\gamma_i]$. Obviously, $\int_z \omega = 0$ for all $z \in H_1(\gamma_i)$, because $\gamma_i \in \mathcal{F}_\omega$. Consequently, on M only integrals over cycles transversal to γ_i can be non-zero: $z_i = D[\gamma_i]$, $i = 1, \dots, N$. Furthermore, the number of integrals $\int_{z_i} \omega$ independent over $\mathbb{Q}$ obviously does not exceed the number of independent classes $[\gamma_i]$, that is, $\operatorname{rk} H_\omega$. Thus, on the manifold M

$$\operatorname{dirr}\omega = \operatorname{rk}_{\mathbb{Q}}\left\{\int_{z_1} \omega, \dots, \int_{z_k} \omega\right\} - 1,$$

where $k = \operatorname{rk} H_\omega$. Consequently, $\operatorname{dirr}\omega \le \operatorname{rk} H_\omega - 1 \le h_0^{\max}(M) - 1$. The theorem is proved.

It is not hard to show that $h_0(M_g^2) = g$, and then Theorem 2 in [6] follows immediately from the theorem proved.

We consider Morse forms in general position.

Corollary. *If on a manifold M the intersection of the $(n-1)$-dimensional homology classes is not identically zero, then the foliation of a form in general position has a non-compact leaf.*

Proof. Since the intersection of the homology classes is not identically zero, it follows that $h_0(M) < \beta_1(M)$. A form in general position has maximal degree of irrationality $\operatorname{dirr}\omega = \beta_1(M) - 1$. Consequently, $\operatorname{dirr}\omega \ge h_0(M)$, and the foliation $\mathcal{F}_\omega$ has a non-compact leaf by the theorem. The corollary is proved.

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Received 8/DEC/94