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Loops in Reeb graphs of n-manifolds

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Abstract The Reeb graph of a smooth function on a connected smooth closed orientable n-manifold is obtained by contracting the connected components of the level sets to points. The number of loops in the Reeb graph is defined as its first Betti number. We describe the set of possible values of the number of loops in the Reeb graph in terms of the co-rank of the fundamental group of the manifold and show that all such values are realized for Morse functions and, except on surfaces, even for simple Morse functions. For surfaces, we describe the set of Morse functions with the number of loops in the Reeb graph equal to the genus of the surface.

Keywords Reeb graph · contour tree · number of loops · Morse function · co-rank of the fundamental group

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1 Introduction

Consider a connected smooth closed orientable n-dimensional manifold M, $n \geq 2$, and a smooth function $f : M \to \mathbb{R}$.

1.1 Basic definitions and motivation

The *Reeb graph* $R(f)$ of f is obtained by contracting the connected components of the level sets $f^{-1}(\text{const})$ to points, and the orientation on $R(f)$ is defined by $\operatorname{grad}(f)$, the gradient of f; see Fig. 1 for an example and Section 3 for details. The Reeb graph of a function shows how the topology of its level sets evolves

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and how they are connected with each other, which is useful for the study of the function's behavior.

The notion of the Reeb graph finds numerous practical applications in computer graphics, geometric model databases, and data visualization. The Reeb graph is also very important for shape analysis in computational topology. A survey of the corresponding techniques can be found in [2].

For some classes of functions, such as functions with isolated critical points, in particular, Morse functions, as well as Morse-Bott functions, the Reeb graph can be thought of as a finite graph endowed with a graph topology. However, the Reeb graph of an arbitrary smooth funcion is a topological space not necessarily homeomorphic to any finite or infinite graph; see Example 6.

1.2 Problem statement

Not every graph can be the Reeb graph of some function [33, Theorem 2.1; 27, Remark 2.3]: for example, a Reeb graph cannot have directed cycles. It can, though, have undirected cycles.[1] In the context of computational geometry, the first Betti number $b_1(R(f))$ of the Reeb graph is called the *number of loops*; see Section 3 for details.

Calculating $b_1(R(f))$ is an important problem. It can be shown that

$$b_1(R(f)) \le b_1(M), \tag{1}$$

where $b_1(M)$ is the first Betti number of the manifold [3, Eq. (1)]. However, this bound is not tight.

Some known results concern $b_1(R(f))$ for specific types of function on a closed orientable surface M_g^2 of genus g. For example, the influential paper [3, Lemma A], to which the title of this paper alludes, shows that for a simple Morse function f (Morse function with only one critical point on each critical level; see Section 2.4 for details) on a closed orientable surface M_g^2 of genus g, it holds

$$b_1(R(f)) = g, \tag{2}$$

which is much stronger than the bound (1) since $b_1(M_g^2)$ is $2g$. There, the authors noted that this equality should be extendable to a wider class of functions [3, Section 2]. Indeed, it has been recently extended to simple Morse-Bott functions [26, Theorem 2]. For smooth functions with isolated critical points, the equality turns into a bound: $0 \le b_1(R(f)) \le g$ [22, Theorem 5.6].

Various authors have noted that $b_1(R(f))$ is related with the structure of the fundamental group $\pi_1(M)$. For instance, in a recent paper Kaluba *et al.* [22, Corollary 5.2], considering functions with isolated critical points on an n-manifold, note that if $\pi_1(M)$ is finite, then $b_1(R(f)) = 0$, i.e., the Reeb graph $R(f)$ is a tree, and show that if the fundamental group $\pi_1(M)$ does not contain F_2, the free group on two generators, then $b_1(R(f))$ is at most 1. This covers discrete groups that are amenable, and in particular, abelian, solvable, or

[1] Some authors refer to a Reeb graph without cycles as a contour tree.

nilpotent. E.g., for n-torus T^n, while $b_1(T^n)$ is n, their result gives an estimate $b_1(R(f)) \leq 1$ much stronger than (1), since $\pi_1(T^n)$ is abelian.

1.3 Contributions

In this paper, we generalize Lemma A from [3], which states the equality (2) for simple Morse functions on closed oriented surfaces, to manifolds of arbitrary dimension and to a wider class of functions (smooth functions), as well as, conversely, describe the class of functions for which this equality holds.

Specifically, we describe the set of possible values of the number of loops in the Reeb graph of an arbitrary smooth function f on a connected closed orientable n-manifold M of arbitrary dimension n in terms of the structure of the fundamental group $\pi_1(M)$:

$$0 \leq b_1(R(f)) \leq b_1'(M), \tag{3}$$

both bounds being tight and all intermediate values being realized by (simple if $n \geq 3$) Morse functions f on a given manifold (see Theorem 13). Here, $b_1'(M)$ stands for corank$(\pi_1(M))$, the co-rank of the fundamental group $\pi_1(M)$, i.e., the maximum rank of its free homomorphic image; see Section 2.3 for details. This important value, which represents the number of cuts of the manifold (genus in the case of a surface), is algorithmically computable [25, Theorem 11.5; 32, Theorem 3]. It can be easily calculated or bounded for manifolds that are connected sums or direct products of simpler manifolds. In particular, in Section 2.3 we present the following properties of the co-rank of the fundamental group:

$$\begin{aligned} b_1'(M_1 \# M_2) &= b_1'(M_1) + b_1'(M_2), \\ b_1'(M_1 \times M_2) &= \max\{\, b_1'(M_1), b_1'(M_2) \,\}. \end{aligned} \tag{4}$$

For example,

$$b_1'(M_g^2) = b_1'\left(\mathop{\#}_{i=1}^{g} (S^1 \times S^1)\right) = \sum_{i=1}^{g} 1 = g. \tag{5}$$

In addition, $b_1'(M)$ is bounded from above by the *isotropy index* $h(M)$ [28, Definition 2], which is the maximum rank of a subgroup in the first cohomology group $H^1(M, \mathbb{Z})$ with trivial cup product

$$\smile : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z});$$

see Section 2.3 for details. The isotropy index $h(M)$ is easier to calculate than the co-rank of the fundamental group $b_1'(M)$: while for the connected sums and direct products of manifolds it is calculated similarly to (4), it can also be bounded in terms of the Betti numbers $b_1(M)$ and $b_2(M)$; see equations (11)–(12).

Since

$$b'_1(M) \leq b_1(M),$$

the bound (3) is stronger than (1): for instance, given an n-torus $T^n = \times_{i=1}^n S^1$, we have $b_1(T^n) = n$, while (4) gives $b'_1(T^n) = 1$.

Our result generalizes various bounds given in [3, Lemma A; 26, Theorem 2; 22, Theorem 5.6] and other papers, whose authors considered only surfaces or selected "good" functions with special types of critical points.

We show, however, that all values satisfying the bounds (3) can indeed be realized by "good" functions, namely, by simple Morse functions, except for surfaces (Theorem 13). Surfaces are an exception because in this case, simple Morse functions satisfy the equality (2), which by (5) corresponds only to the upper bound in (3). Still all non-negative values of $b_1(R(f))$ smaller than g are realized on M_g^2 by (non-simple) Morse functions. As an example, for any surface we explicitly construct a (simple when possible) Morse function with any given number of loops $b_1(R(f))$ within the bounds of (3) (Example 18).

Conversely, we describe the set of Morse functions on an orientable surface M_g^2 of genus g that satisfy the equality (2): these are those functions for which small regular neighborhoods of singular level sets have genus zero (Theorem 9). We call such functions as *topologically simple* Morse functions (Definition 8); see Section 4 for details. This notion is a generalization of the notion of simple Morse function (Theorem 10). It is specific for surfaces that for simple Morse functions the number of loops $b_1(R(f))$ in the Reeb graph does not depend on the function—as we show in Theorem 13, for n-manifolds with $n \geq 3$, it can take any value within the bounds of (3).

Technically, our results are based on the important fact that the number of loops $b_1(R(f))$ is equal to $c(df)$, the number of homologically independent compact leaves of the foliation defined by the function f (Theorem 7). This allows using the well-studied theory of Morse form foliations, where the value $c(\omega)$ for a Morse form ω (locally the differential of a Morse function) plays an important role and has been extensively investigated by homological, group-theoretic, and graph-theoretic methods [8,13]. In particular, the bounds in (3) hold for it and all intermediate values are known to be realized by Morse forms (Theorem 3). In this paper we show that they are realized by exact Morse forms df, where f can be chosen to be a simple Morse function.

The paper is structured as follows. In Section 2, we introduce the necessary notions and facts about closed one-form foliations, the co-rank of the fundamental group, including some methods for its calculation, and Morse functions. In Section 3, we show that the number of loops in the Reeb graph coincides with the number of homologically independent compact leaves of the foliation defined by the function. In Section 4, we describe the class of Morse functions on a closed orientable surface for which the number of loops in the Reeb graph coincides with the surface's genus. In Section 5, we prove our main result: tight bounds on the number of loops in the Reeb graph in terms of the co-rank of the fundamental group. We show that all values within these bounds are possible for Morse functions and, except for surfaces, even for simple Morse functions.

Finally, in Section 6 we discuss open problems and further directions of this work.

2 Definitions and useful facts

We consider a smooth closed connected orientable n-dimensional manifold M with $n \geq 2$.

2.1 Closed one-form foliation

Consider a closed one-form ω on M, and denote by $\operatorname{Sing}\omega$ its singular set: $\operatorname{Sing}\omega = \{ p \in M \mid \omega(p) = 0 \}$. This form defines a foliation $\mathcal{F}_\omega$ on the complement of the singulat set. A leaf γ of the foliation $\mathcal{F}_\omega$ can be *compactifiable* ($\gamma \cup \operatorname{Sing}\omega$ is compact) or *non-compactifiable*. In particular, compact leaves are compactifiable. A foliation is called *compactifiable* if all its leaves are compactifiable.

Consider also a smooth function $f: M \to \mathbb{R}$. Its critical set $\operatorname{Crit}(f)$ is non-empty. On the complement of this set, the function defines a foliation $\mathcal{F}_f$, whose leaves are connected components of level sets $\{ f(x) = \text{const} \}$. Compact leaves are those connected components of (possibly critical) level sets that do not contain critical points. This foliation coincides with the foliation of the exact form df.

Compact leaves are submanifolds of M. We denote by $c(\omega)$ the number of homologically independent compact leaves of the foliation $\mathcal{F}_\omega$. In particular,

$$c(\omega) \leq b_1(M), \tag{6}$$

the first Betti number. In the case when ω is the exact form df, we denote $c(df)$ simply by $c(f)$.

For any integer number c, the set of forms such that $c(\omega) \geq c$ is open in the space of closed one-forms on M of a given cohomology class [13, Theorem 3.1].

2.2 Foliation graph

For simplicity, in this section we assume that the set $\operatorname{Sing}\omega$ of singularities of ω is non-empty, though the facts presented here can be meaningfully generalized to non-singular forms.

The union $\mathcal{C}$ of all compact leaves of a closed one-form foliation is open [13, Lemma 3.1]. Its connected components $\mathcal{C}_i$ are called *maximal components*. Each maximal component $\mathcal{C}_i$ is a cylinder over a compact leaf γ_i from $\mathcal{C}_i$:

$$\mathcal{C}_i \cong \gamma_i \times (0, 1), \tag{7}$$

where the diffeomorphism maps $\gamma_i \times t$ to leaves of the foliation $\mathcal{F}_\omega$. Denote by Δ the complement of $\mathcal{C}$. Obviously, the boundaries $\partial\mathcal{C}_i$ lie in Δ. Since each maximal component is a cylinder, it adjoins one or two connected components Δ_j of Δ.

We obtain a decomposition of the manifold into mutually disjoint sets:

$$M = \bigcup_i \mathcal{C}_i \cup \bigcup_j \Delta_j. \tag{8}$$

This decomposition allows representing the manifold M as a pseudograph[2] Γ_ω, called the *foliation graph*, with maximal components $\mathcal{C}_i$ as edges and the sets Δ_j as vertices [7, Section 2]. In this graph, an edge $\mathcal{C}_i$ is incident to a vertex Δ_j if and only if the boundary $\partial\mathcal{C}_i$ instersects Δ_j. The foliation graph can be directed by the gradient of f (locally $\omega = df$). We denote the directed foliation graph by $\mathbf{\Gamma}_\omega$. In the case of an exact form, $\omega = df$, we denote Γ_{df} by Γ_f and $\mathbf{\Gamma}_{df}$ by $\mathbf{\Gamma}_f$; see Figure 1.

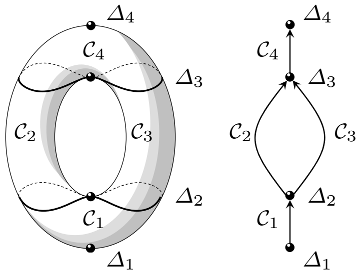

Fig. 1 Decomposition (8) for a height function f on a torus T^2 and the corresponding foliation graph $\mathbf{\Gamma}_f$ or the Reeb graph $R(f)$. Figure adapted from [7, Fig. 1].

2.3 Co-rank of the fundamental group

The *co-rank* corank(G) of a finitely generated group G is the maximum rank of a free quotient group of G, i.e., the maximum rank of a free group F such that there exists an epimorphism $\varphi\colon G \twoheadrightarrow F$ [21, page 411; 24, page 37]. Since the manifold M is compact, its fundamental group $\pi_1(M)$ is finitely presented and thus has a co-rank. We will denote the co-rank of the fundamental group of M, also called the first non-commutative Betti number [1, page 142], by $b_1'(M)$.

This value represents the number of cuts of the manifold (genus in the case of a surface): on the manifold M there exist at most $b_1'(M)$ mutually disjoint and homologically independent closed codimension-one submanifolds [21, Theorem 2.1]. In particular, for any closed one-form ω, we have

$$c(\omega) \le b_1'(M), \tag{9}$$

[2] A graph admitting multiple edges and loops, i.e., edges that connect a vertex with itself.

which is stronger than (6).

Though co-rank is known to be algorithmically computable for finitely presented groups [25, Theorem 11.5; 32, Theorem 3], we are not aware of any simple method of finding $b_1'(M)$ for a given manifold. However, for a smooth closed connected manifold this value is bounded from above by [4, Eq. (9)]

$$b_1'(M) \le h(M) \le b_1(M),$$

where the *isotropy index* $h(M)$ [28, Definition 2] is the maximum rank of a subgroup of the first cohomology group $H^1(M,\mathbb{Z})$ on which the cup-product

$$\smile : H^1(M,\mathbb{Z}) \times H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z}) \tag{10}$$

is trivial. The isotropy index is often simpler to calculate than $b_1'(M)$.

For some simple or low-dimensional manifolds, such as closed orientable surface $M_g^2 = \#^g\, T^2$ (connected sum) of genus g and closed non-orientable surface $N_h^2 = \#^h\, \mathbb{R}P^2$, $h \ge 1$, these values are obvious or well known:

circle:	$b_1'(S^1) = h(S^1) = 1,$	
sphere, $n \ge 2$:	$b_1'(S^n) = h(S^n) = 0,$	
n-torus:	$b_1'(T^n) = h(T^n) = 1,$	[14, Exmp. 3.5; 29, Exmp. 1]
orientable surface:	$b_1'(M_g^2) = h(M_g^2) = g,$	[24, Lemma 2.1; 29, Exmp. 2]
non-orientable surface:	$b_1'(N_h^2) = \left[\frac{h}{2}\right],\ h(N_h^2) = h-1.$	[14, Eq. (4.1)]

For the connected sum of connected closed n-manifolds, $n \ge 2$, except for non-orientable surface N_h^2, and for the direct product of connected closed manifolds it holds

$$b_1'(M_1 \# M_2) = b_1'(M_1) + b_1'(M_2), \qquad [14, \text{Eq. } (1.1)]$$
$$b_1'(M_1 \times M_2) = \max\{\, b_1'(M_1), b_1'(M_2)\,\}, \qquad [14, \text{Theorem } 3.1]$$

and similar equalities hold for $h(M)$ [15, Theorems 21 and 27] (in the case of the connected sum, for orientable manifolds). One can see that for many manifolds $b_1'(M)$ and $h(M)$ coincide. In particular, this is the case if the fundamental group $\pi_1(M)$ is a quasi-Kähler 1-formal group [4, Theorem 1.2]. However, there exist manifolds with $b_1'(M) < h(M)$, for example, the Heisenberg nilmanifold H^3: since the fundamental group $\pi_1(H^3)$ is nilpotent, $b_1'(H^3)$ equals 1, whereas $h(H^3)$ equals 2 [23, Section 6.1].

The isotropy index $h(M)$ in turn can be estimated via the Betti numbers $b_1(M)$ and $b_2(M)$ and the structure of the cup product (10). Namely, denoting $b_i(M)$ by b_i and $\operatorname{rk}\ker \smile$ by k, for a smooth closed connected orientable n-manifold with $n \ge 2$, we have [29, page 5]

$$\frac{b_1 + k b_2}{b_2 + 1} \le h(M) \le \frac{b_1 b_2 + k}{b_2 + 1}, \tag{11}$$

which reduces to $h(M) = \frac{1}{2}(b_1 + k)$ in the case of $b_2 = 1$ and to $h(M) = b_1(M)$ in the case of $b_2 = 0$. If the cup-product $\smile$ is surjective, then

$$h(M) \leq k + \frac{1}{2} + \sqrt{\left(b_1 - k - \frac{1}{2}\right)^2 - 2b_2}. \tag{12}$$

In addition to the isotropy index over integers, $h(M) = h(M; \mathbb{Z})$, one can define the isotropy index over a field, $h(M; F)$, via the first cohomology group $H^1(M; F)$ with coefficients in the field F. If the field is of characteristic zero, then $h(M) \leq h(M; F)$; in particular, $h(M; \mathbb{Q}) = h(M)$ [15, Lemma 10]. For a field, $h(M; F)$ can be conveniently calculated using vector spaces instead of groups.

2.4 Morse functions and Morse forms

A smooth function $f : M \to \mathbb{R}$ is called a *Morse function* if all its critical points are non-degenerate. Then this set is finite since the critical points are isolated and M is compact.

Proposition 1 ([18, Theorem 6.2 of Chapter II]) *The set of Morse functions is open and dense in the space $C^\infty(M, \mathbb{R})$ of all smooth functions on a given smooth manifold.*

A Morse function is *simple* (called also *nonresonant* or *excellent*) if each critical level contains exactly one critical point.

Proposition 2 ([30, Proposition 1.29]) *The set of simple Morse functions is open and dense in the space of all Morse functions on a given smooth manifold.*

A closed one-form ω on M is called a *Morse form* if it is locally the differential of a Morse function. The set of its singularities is finite, since the singularities are isolated and M is compact. The set of Morse forms is open and dense in the space of closed one-forms on M [13, Lemma 2.1].

A Morse form with compactifiable foliation is called *generic* if each set Δ_i in (8) contains exactly one singularity [6, Definition 9.1]. Then, for a simple Morse function f, the form df is generic.

Theorem 3 ([8, Theorem 8, Remark 12]) *Let c be an integer number. Then there exists a Morse form ω on M such that $c(\omega) = c$ if and only if*

$$0 \leq c \leq b'_1(M),$$

where $c(\omega)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_\omega$ and $b'_1(M)$ is the co-rank of the fundamental group. The form can be chosen with compactifiable foliation and, if $\dim M \geq 3$, generic.

In this paper we show that, moreover, the form ω can be chosen exact, and, if $\dim M \geq 3$, can even be chosen to be df for a simple Morse function f.

Denote the *form's rank* over $\mathbb{Q}$, $\operatorname{rk}\operatorname{im}[\omega]$, where $[\omega]\colon H_1(M) \to \mathbb{R}$ is the integration map, by $\operatorname{rk}\omega$. Obviously, $0 \leq \operatorname{rk}\omega \leq b_1(M)$, and for an exact form $\omega = df$, we have $\operatorname{rk}\omega = 0$.

A given foliation $\mathcal{F}_\omega$ can be defined by different forms (called *collinear forms* [10, Definition 3.1]) and even by forms of different ranks:

Theorem 4 ([11, Eq. (14) or Theorem 4.11, first case]) *Let $\mathcal{F}_\omega$ be a compactifiable Morse form foliation and r be an integer number. Then there exists a Morse form ω' such that $\mathcal{F}_{\omega'} = \mathcal{F}_\omega$ and $\operatorname{rk}\omega' = r$ if and only if*

$$a \leq r \leq c(\omega),$$

where $\operatorname{rk}\omega$ is the form's rank, $c(\omega)$ is the number of homologically independent compact leaves of $\mathcal{F}_\omega$, and

$$a = \begin{cases} 0 & \text{if } \mathbf{\Gamma}_\omega \text{ has no directed cycles,} \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathbf{\Gamma}_\omega$ is the directed foliation graph.

Theorem 5 ([9, Theorem 2.1]) *Let ω be a Morse form on M. Then*

$$m(\Gamma_\omega) = c(\omega),$$

where $m(\Gamma_\omega)$ is the circuit rank[3] *of the foliation graph Γ_ω and $c(\omega)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_\omega$.*

Example 6 below shows that the condition for the form to be of Morse type is important. In fact it would be sufficient to require for Γ_ω to be finite.

3 Reeb graph

Given a topological space X and a continuous function $f\colon X \to \mathbb{R}$, consider an equivalence relation $\sim$ on X, where $p \sim q$ whenever p and q belong to the same connected component of a level set $f^{-1}(\text{const})$ (called by some authors *contour*). The *Reeb graph* $R(f)$ is the quotient space $X/{\sim}$, endowed with the quotient topology. Since in our case $X = M$, a compact connected manifold, and the quotient map $\varphi\colon M \to R(f)$ is continuous, the Reeb graph $R(f)$ is a path-connected Hausdorff space. Fig. 1 can be considered as a simple example of a Reeb graph.

By the *number of loops* in the Reeb graph we understand its first Betti number $b_1(R(f))$, which in our case coincides with the rank of its fundamental group [17].

[3] The number of independent cycles in the graph, also called *cyclomatic number* or *nullity.*

The Reeb graph of a smooth function on M could be thought of as a graph in the graph-theoretic sense, the maximal components being its edges and other points being vertices; this graph can be shown to be isomorphic to the foliation graph Γ_f. However, discarding the topology of the Reeb graph results in losing important properties:

Example 6 Consider the 2-torus $T^2 = S^1_1 \times S^1_2$ with a coordinate system (x, t), $x \in S^1_1$, $t \in S^1_2$. Consider a function $f : T^2 \to \mathbb{R}$ given by $f(x, t) = e^{-\frac{1}{t^2}} \cos(\frac{1}{t})$ on an interval $t \in (-\varepsilon, \varepsilon)$ and glued smoothly outside of this interval; see Fig. 2. Then $Crit(f)$ has an infinite number of connected components $\Delta_i = S^1$, and thus the foliation has an infinite number of maximal components $\mathcal{C}_i$. The Reeb graph $R(f)$ is S^1. In particular, it is connected, and for the number of loops, we have $b_1(R(f)) = c(f) = 1$. However, the foliation graph Γ_f is infinite and, considered without any topology, is not connected: indeed, the vertex $\{ t = 0 \}$ is not connected by an edge with any other vertex and thus forms a connected component by itself. In particular, for the circuit rank of the foliation graph, we have $m(\Gamma_f) = 0 < c(f)$.

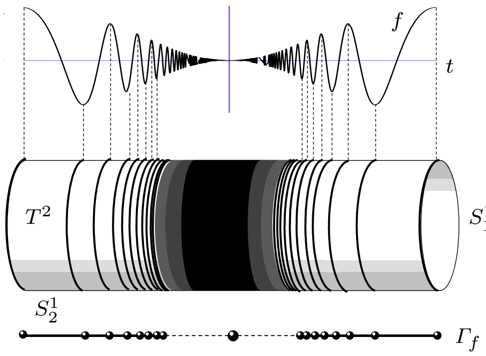

Fig. 2 A smooth function f on a 2-torus T^2 shown as a cylinder with identified sides. While its Reeb graph (shown at the bottom as a line with identified ends) is connected as a topological space and has $b_1(R(f)) = c(f) = 1$, its foliation graph Γ_f is not connected: the vertex in the center is not connected by an edge to any other node. In particular, $0 = m(\Gamma_f) \neq c(f) = 1$.

Theorem 7 *Let f be a smooth function on M with the Reeb graph $R(f)$. Then*

$$b_1(R(f)) = c(f),$$

where $b_1(R(f))$ is the number of loops in the Reeb graph and $c(f)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_f$.

Proof Consider the foliation $\mathcal{F}_f$. By [16, Theorem 3.1], the group generated by the homology classes of all compact leaves has a basis consisting of homology classes of compact leaves $[\gamma_1], \ldots, [\gamma_c]$, where $c = c(f)$.

The quotient map φ maps each leaf γ_i to a point in the Reeb graph $R(f)$ that by (7) has a neighborhood homeomorphic to $\mathbb{R}$. Consider

$$T = R(f) \setminus \bigcup_i^c \varphi(\gamma_i) = \varphi\left(M \setminus \bigcup_i^c (\gamma_i)\right).$$

Since the leaves γ_i are homologically independent, the latter manifold is connected. As the Reeb graph of a function on a connected compact manifold with boundary such that f is constant at the boundary and the complement of any non-boundary non-singular leaf of f is not connected, T is a dendrite [17]. We have obtained c cuttings of the Reeb graph $R(f)$ that leave it connected but result in a homologically trivial space. Thus $b_1(R(f)) = c$ [17]. □

4 Reeb graph of a Morse function on a surface

Recall that a *simple Morse function* is a Morse function with a unique critical point on each critical level. For a simple Morse function on a closed orientable surface M_g^2 of genus g, it has been shown in [3, Lemma A] that the number of loops in the Reeb graph, $b_1(R(f))$, coincides with the genus g. In this section, we completely characterize the subclass of Morse functions satisfying this property, which we call *topologically simple* Morse functions.

4.1 Morse functions on a surface

Consider a Morse function f on a closed orientable surface M_g^2 of genus g. Denote by Δ_i those connected components of level sets of f that contain a critical point. These sets can be shown to coincide with Δ_i from (8), the vertices of the foliation graph Γ_f. Following [6, Section 9.1], we will call them *singular leaves.* A Morse function has a finite number of critical points and thus a finite number of singular leaves, which are finite subcomplices of M_g^2.

For a subset X of M, its *regular neighborhood* $V(X)$ is a locally flat, compact submanifold of M, which is a topological neighborhood of X, such that the inclusion $X \hookrightarrow V(X)$ is a simple homotopy equivalence and X is a strong deformation retract of $V(X)$ [31, Definition 1]. Since a singular leaf Δ_i of a Morse function can be viewed as a finite CW-complex, it has a regular neighborhood $V(\Delta_i)$ [19, Theorem 1].

The *genus* $g(V)$ of an orientable surface V is the maximum number of cuttings along closed simple curves without increasing the number of its connected components. For a closed orientable surface M_g^2 given by the connected sum of g tori, we have $g(M_g^2) = g$. For a compact surface $V \subset M_g^2$, it holds $g(V) \leq g$ ([12, Corollary 8]).

For a Morse function f on M_g^2, it holds [12, Theorem 42]

$$c(f) = g - \sum_i g(V(\Delta_i)), \tag{13}$$

where $c(f)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_f$, $V(\cdot)$ is a small regular neighborhood, and $g(\cdot)$ is the genus of a surface. The summation is taken over all singular leaves Δ_i.

Denote by $d(\Delta_i)$ the number of maximal components glued to the singular leaf Δ_i, i.e., the degree of Δ_i as a vertex of the foliation graph Γ_f, and by Ω_1 the set of critical points of index 1 (saddles) of f. Then [12, Lemma 29]

$$g(V(\Delta_i)) = 1 + \frac{1}{2}(|\Delta_i \cap \Omega_1| - d(\Delta_i)). \tag{14}$$

4.2 Morse functions with number of loops in the Reeb graph equal to genus

We can generalize the notion of a simple Morse function as follows:

Definition 8 A Morse function f is *topologically simple* if the regular neighborhood of each its singular leaf Δ_i has genus zero.

Theorem 9 *Let M_g^2 be a smooth closed connected orientable surface of genus g and $f: M_g^2 \to \mathbb{R}$ a Morse function. Then the number of loops in the Reeb graph $b_1(R(f))$ is equal to g if and only if f is a topologically simple Morse function.*

Proof Theorem 7 gives $b_1(R(f)) = c(f)$, while (13) implies that $c(f)$ equals g if and only if f is topologically simple. □

Theorem 10 *A simple Morse function is topologically simple.*

While this fact is already implied by (2), for completeness we will give an independent proof. For this, we need two lemmas.

Lemma 11 *Let f be a Morse function and Δ_i its singular leaf. Then Δ_i has a small regular neighborhood of genus zero if and only if $d(\Delta_i) = |\Delta_i \cap \Omega_1| + 2$ or $d(\Delta_i) = 0$ (in the latter case Δ_i consists of a single point, which is an extremum).*

In particular, for a topologically simple Morse function, each singular leaf Δ_i with k saddles adjoins exactly $k + 2$ maximal components; see Figure 3. The proof follows from (14).

Lemma 12 *Let f be a Morse function. If a singular leaf Δ_i contains exactly one saddle, then $d(\Delta_i) = 3$.*

Proof By (14), if $|\Delta_i \cap \Omega_1| = 1$, then $d(\Delta_i) \leq 3$. If $g(V(\Delta_i)) = 0$, then by Lemma 11 we have $d(\Delta_i) = 3$. Suppose $g(V(\Delta_i)) \geq 1$. Then by (14), $d(\Delta_i) \leq 1$. Since for a Morse function, each singularity must adjoin at least one maximal component, $d(\Delta_i) \neq 0$, thus $d(\Delta_i) = 1$, i.e., there is only one maximal component $\mathcal{C}_j$ adjacent to Δ_i from both below and above, and thus this $\mathcal{C}_j$ is adjacent only to this Δ_i. Then $\mathcal{C}_j \cup \Delta_i$ is a closed connected surface M_k^2 with only one critical point. However, its Euler characteristic $\chi(M_k^2) = 2 - 2k$ is even, a contradiction. □

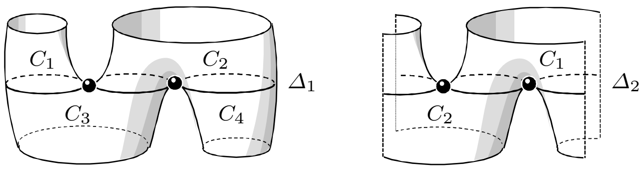

Fig. 3 Both Δ_i have two singularities, but their regular neighborhoods have different topological type. Left: $g(V(\Delta_1)) = 0$ and Δ_1 adjoins four maximal components $\mathcal{C}_i$. Here, $d(\Delta_1) = 4$, the degree in the foliation graph. Right: the vertical dotted lines are pairwise identified. Here, $g(V(\Delta_2)) = 1$ and Δ_2 adjoins two maximal components $\mathcal{C}_i$, with $d(\Delta_2) = 2$.

Proof (of Theorem 10) Let f be a simple Morse function, i.e., each its singular leaf Δ_i contains exactly one critical point. Then the proof follows from Lemmas 12 and (14). □

5 Main theorem: Number of loops in the Reeb graph

Our main result consists in the following theorem:

Theorem 13 *Let M be a smooth closed connected orientable manifold, and m an integer. Then there exists a smooth function $f: M \to \mathbb{R}$ such that the number of loops $b_1(R(f))$ (first Betti number) in its Reeb graph is equal to m if and only if*

$$0 \le m \le b_1'(M),$$

where $b_1'(M)$ is the co-rank of the fundamental group $\pi_1(M)$.

The function f can be chosen to be a Morse function. Moreover, f can be chosen to be a simple Morse function if and only if either $\dim M \ge 3$ *or M is a surface of genus m.*

For the proof, we will need some lemmas.

Lemma 14 *Let M be a smooth closed manifold, $\gamma \subset M$ a codimension-one submanifold, and $C \cong \gamma \times (0,1) \subset M$ a cylindrical subset with a coordinate system (x,t), $x \in \gamma$, $t \in (0,1)$. Let $\varphi: \mathbb{R} \to \mathbb{R}$ be a Morse function with* $\mathrm{Crit}(\varphi) \subset (0,1)$ *and $\Phi: C \to \mathbb{R}$ a function defined by $\Phi(x,t) = \varphi(t)$. Let ω be a closed one-form on M with only Morse singularities on $M \setminus C$ such that*

$$\omega|_C = d\Phi. \tag{15}$$

Denote by U a small neighborhood of $\mathrm{Sing}\,\omega \cap C$. *Then there exists a Morse form ω' on M such that*

$$\omega'|_{M\setminus U} \equiv \omega. \tag{16}$$

If, in addition, ω is generic on $M \setminus C$, then ω' can be chosen generic.

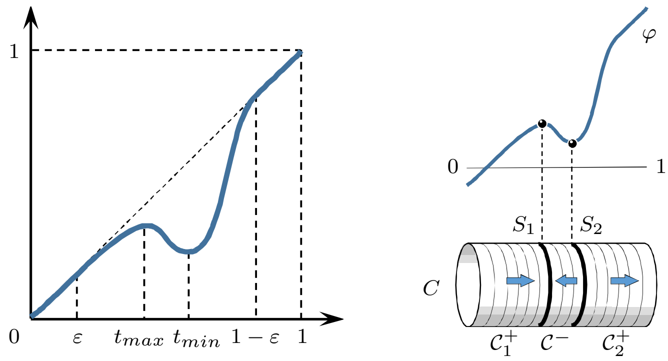

Fig. 4 Left: the Morse function φ, $\varphi(t) = t$ on $\mathbb{R} \setminus (\varepsilon, 1-\varepsilon)$, with one maximum and one minimum in $(0, 1)$. Right: the form $d\varphi(t)$ on a maximal component C has non-Morse singular subsets $S_1 = \gamma \times \{t_{max}\}$ and $S_2 = \gamma \times \{t_{min}\}$ and three maximal components $\mathcal{C}_1^+$, $\mathcal{C}^-$, and $\mathcal{C}_2^+$ with alternating directions of the corresponding arcs in the directed foliation graph. The second summand in (19) slightly perturbs this function a small neighborhood U of S^1 and S^2 to produce a Morse function that has the same maximal components outside U.

Proof The form ω has non-Morse critical subset $\operatorname{Sing}\omega \cap C = \gamma \times \operatorname{Crit}(\varphi)$ for a finite set $\operatorname{Crit}(\varphi) \subset (0, 1)$; see Fig. 4. Since γ is compact, there exists a small neighborhood V of $\operatorname{Crit}(\varphi)$ such that $\overline{V} \subset (0, 1)$ and $\gamma \times V \subset U$.

Consider a smooth function $\psi : \mathbb{R} \to \mathbb{R}$ satisfying

$$\psi(t) = \begin{cases} \varphi(t) & \text{in } W, \quad (17) \\ 0 & \text{in } \mathbb{R} \setminus V, \quad (18) \end{cases}$$

where $W \subset V$ is a smaller neighborhood of $\operatorname{Crit}(\varphi)$. By Proposition 1, there exists a Morse function $f = f(x)$ on γ. On C, consider a function

$$F(x, t) = \varphi(t) + \lambda\psi(t)f(x) \tag{19}$$

for a small enough λ. Then the form

$$\omega' = \begin{cases} dF & \text{in } C, \\ \omega & \text{in } M \setminus C \end{cases}$$

has the desired properties. Indeed, by (15) and (18), the form ω is smooth and satisfies (16). We only need to show that all critical points of F are of Morse type.

On $C \setminus (\gamma \times W)$ from (17), for a small enough λ we have

$$\frac{\partial F}{\partial t} = \dot{\varphi} + \lambda f \dot{\psi} \neq 0,$$

since both f and $\dot{\psi}$ are bounded and $|\dot{\varphi}| > \text{const} > 0$. Therefore $\operatorname{Crit}(F) \subseteq \gamma \times W$, where by (17),

$$F(x, t) = \varphi(t)(1 + \lambda f(x)).$$

Since the two factors, which can be assumed non-vanishing, depend on different variables, we have

$$\mathrm{Crit}(F) = \mathrm{Crit}(f) \times \mathrm{Crit}(\varphi) \tag{20}$$

and the critical points are non-degenerate since so are those of the factors.

Let now ω be generic on $M \setminus C$. We can assume that each connected component W_1 of W from (17) contains only one critical point of φ. By Proposition 2, f can be chosen to be a simple Morse function. Then, by (20), so is F in $\gamma \times W_1$, and thus the form $\omega' = dF$ is generic in $\gamma \times W_1$. Since on C, the Δ_i from (8) of the form ω' lie in the connected components of $\gamma \times W$, we obtain that ω' is generic. □

Lemma 15 *Let M be a smooth closed connected orientable manifold, $\gamma \subset M$ a codimension-one submanifold, and $C \cong \gamma \times (0, 1) \subset M$ a cylindrical subset with a coordinate system (x, t), $x \in \gamma$, $t \in (0, 1)$. Let ω be a Morse form on M such that the levels $\{\, t = const \,\}$ are leaves of the foliation $\mathcal{F}_\omega$ when $t \in (0, \varepsilon] \cup [1 - \varepsilon, 1)$ for some small ε. Then the subgraph $\mathit{\Gamma}_\omega^C$ of the foliation graph $\mathit{\Gamma}_\omega$ in C is acyclic.*

Proof Since $\{\, t = const \,\}$ are leaves of $\mathcal{F}_\omega$ near the ends of the cylinder, for each leaf L of $\mathcal{F}_\omega$ we have $L \subset C$ if $L \cap C \neq \emptyset$. In particular, the subgraph $\mathit{\Gamma}_\omega^C$ is well-defined. Suppose there is a cycle z in $\mathit{\Gamma}_\omega^C$. Then there exists a leaf L of $\mathcal{F}_\omega$, $L \subset C$, transversal to z, i.e., $[z] \cdot [L] \neq 0$. However, $C = \gamma \times (0, 1)$ is a cylinder, so k-cycles in C are induced from γ for all k. Therefore $[z] \in H_1(\gamma)$ and $[L] \in H_{n-1}(\gamma)$. This implies $[z] \cdot [L] = 0$, a contradiction. □

Lemma 16 *Let M be a smooth closed connected orientable manifold and ω a Morse form on M. Then there exists a Morse form ω' on M with the same circuit rank of the foliation graph, $m(\mathit{\Gamma}_{\omega'}) = m(\mathit{\Gamma}_\omega)$, such that $\mathbf{\Gamma}_{\omega'}$ has no directed cycles. If ω is generic, then ω' can be chosen generic.*

Proof If ω has no maximal components, then it has the desired properties.

Consider a maximal component $\mathcal{C}_i \subset M$. Without loss of generality we assume that $\mathrm{Sing}\,\omega$ is not empty—otherwise we can locally perturb the form to obtain a local maximum. Then by (7) the maximal component is an open cylinder $\mathcal{C}_i \cong \gamma \times (0, 1)$ over a leaf γ of $\mathcal{F}_\omega$ with coordinates (x, t), where $x \in \gamma$ and $t \in (0, 1)$. We can assume $\omega = dt$, so that the foliation on $\mathcal{C}_i$ is defined by $t = \mathrm{const}$.

Consider a Morse function $\varphi \colon \mathbb{R} \to \mathbb{R}$ with one maximum t_{max} and one minimum t_{min}, such that $\varphi(t) = t$ on $\mathbb{R} \setminus (\varepsilon, 1 - \varepsilon)$ for a small $\varepsilon > 0$; see Fig. 4, left. Consider a function $F(x, t) = \varphi(t)$ on $\mathcal{C}_i$ and a closed one-form

$$\omega_i = \begin{cases} dF & \text{on } \mathcal{C}_i, \\ \omega & \text{on } M \setminus \mathcal{C}_i, \end{cases}$$

which is smooth because $dF \equiv dt \equiv \omega$ near the boundaries of $\mathcal{C}_i$.

The foliation graph $\mathbf{\Gamma}_{\omega_i}$ is obtained from $\mathbf{\Gamma}_\omega$ by inserting in the middle of the edge $\mathcal{C}_i$ two new vertices and an edge with the opposite direction; see Fig. 4, right. This operation preserves the circuit rank but breaks any directed cycle that passes by $\mathcal{C}_i$.

By Lemma 14, there exists a Morse form ω_i' that differs from ω_i only in a small neighborhood of the two inserted vertices. In particular, the subgraph of $\mathbf{\Gamma}_{\omega_i'}$ that substitutes the edge $\mathcal{C}_i$ of $\mathbf{\Gamma}_{\omega_i}$ still contains a bridge edge with the opposite direction. By Lemma 15, the inserted subgraph is acyclic, so still $m(\Gamma_{\omega_i'}) = m(\Gamma_\omega)$.

Repeating the process for all maximal components $\mathcal{C}_i$ (there is a finite number of them), we obtain a Morse form ω' on M with $m(\Gamma_{\omega'}) = m(\Gamma_\omega)$ but without directed cycles in $\mathbf{\Gamma}_{\omega'}$.

Moreover, if ω is generic, then by Lemma 14 the form ω_i' above can be chosen generic, which gives a generic ω'. □

Lemma 17 *Let M be a smooth closed connected orientable manifold and ω an exact generic Morse form on M. The set of simple Morse functions is open and dense in the set of Morse functions on M that define the foliation $\mathcal{F}_\omega$.*

Proof Let f be a function such that $\omega = df$. Consider a function φ similar to the one shown in Fig. 4, left, but this time monotonous. Applying it to a maximal component $\mathcal{C}_j$ of ω as in Lemma 16 results in a Morse function in Fig. 4, right, with the same foliation but different integral by $\mathcal{C}_j$. This allows slight variation of the integral of ω by a maximal component, preserving $\mathcal{F}_f$.

Consider a connected component Δ_i of a critical level set of f. It contains only one critical point. By the Sard lemma, we can choose an arbitrary small ε such that $f(\Delta_i) + \varepsilon$ is a regular level. Increase by ε the integral of the form $\omega = df$ by each maximal component $\mathcal{C}_j$ of $\mathcal{F}_\omega$ that represents an incoming edge of Δ_i, by a slight perturbation of f in the middle of $\mathcal{C}_i$ that leaves it constant on leaves of $\mathcal{F}_\omega$. Similarly, decrease by ε the integral by each outgoing edge. This increases $f(\Delta_i)$ by ε, making its critical point unique on its critical level, but preserving exactness of the form and leaving the foliation $\mathcal{F}_\omega$ intact.

Repeating this process for each Δ_i (there is a finite number of them) results in a simple Morse funtion. □

Proof (of Theorem 13) By Theorem 7 and the bound (9), we have

$$0 \leq b_1(R(f)) = c(f) \leq b_1'(M),$$

where $c(f)$ is the number of homologically independent compact leaves of the foliation $\mathcal{F}_f$.

Now consider an integer $0 \leq m \leq b_1'(M)$. By Theorems 3 and 5, on M there exists a Morse form ω defining a compactifiable foliation with $c(\omega) = m = m(\Gamma_\omega)$. By Lemma 16, we can assume Γ_ω to have no directed cycles, and thus by Theorem 4, the form can be assumed exact: $\omega = df$ for a Morse function f.

If $\dim M \geq 3$, in the previous paragraph the form ω from Theorem 3 can be chosen generic, which is preserved by Lemma 16, as well as by Theorem 4 since it preserves $\mathcal{F}_\omega$. Then by Lemma 17, f can be chosen to be simple.

If $\dim M = 2$, i.e., M is a surface of genus g, then by Theorem 9, $m = g$ if and only if f is a topologically simple Morse function. By Theorem 10, this includes the case of simple Morse functions. Existence of a simple Morse function on M follows from [6, Lemma 9.2]. □

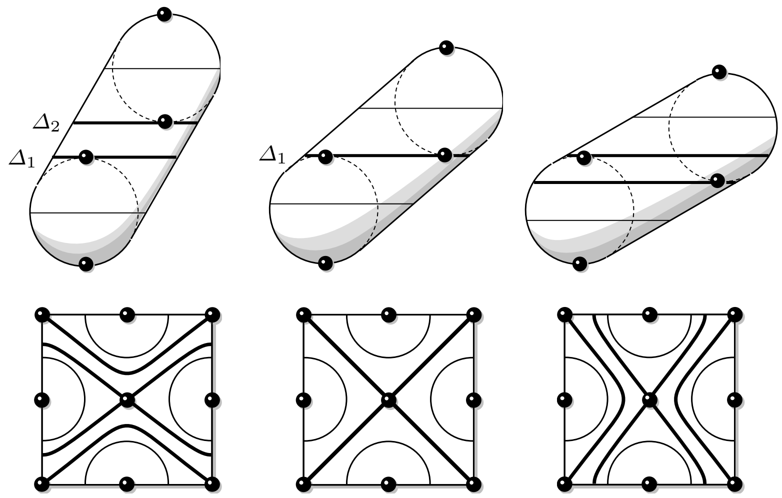

Fig. 5 Top row: side view of a torus with height function as in Fig. 1 but tilted by different angles. Critical levels are shown in bold. Bottom row: the corresponding foliations on the torus represented as a square with pairwise identified sides. On the left and right, the height function is a simple Morse function, but in the middle it is not.

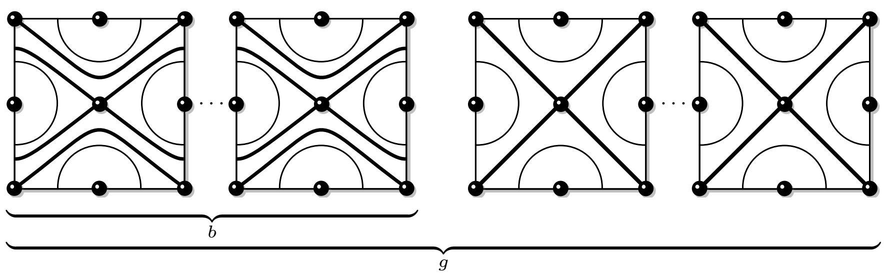

Fig. 6 Surface of genus g constructed as connected sum of b tori with a simple Morse height function as on the left of Fig. 5 and $g - b$ tori with a degenerated height function as in the middle of Fig. 5. This is a simple Morse function if and only if $b = g$.

Example 18 On a closed connected orientable surface M_g^2 of genus g, a (simple) Morse function f with a given number of loops in the Reeb graph $b_1(R(f))$ can be explicitly constructed as follows. Since $b_1'(M_g^2) = g$ [24, Lemma 2.1; 20, Corollary 3.3], by Theorem 13 we have

$$0 \le b_1(R(f)) \le g.$$

Consider an integer $0 \le b \le g$. The surface $M_g^2 = \#_{i=1}^g T_i^2$ is the connected sum of g tori. Consider two types of foliation on T^2: (i) the one defined by a usual height function, Fig. 5, left, and (ii) the one defined by a height function on the torus tilted by a specific angle, with two critical points on a critical level, Fig. 5, middle. The foliation defined by the former function has homologically non-trivial compact leaves, $c(f) = 1$. The foliation defined by the latter function has no homologically non-trivial compact leaves, $c(f) = 0$. However, its critical level Δ_1 has a regular neighborhood $V(\gamma)$ with $g(V(\gamma)) = 1$. Choosing on b tori T_i^2 the function of the first type and on $g-b$ tori the function of the second type gives a function f on M_g^2 with $c(f) = b_1(R(f)) = b$; see Fig. 6. Note that in accordance with Theorem 13, the result is a simple Morse function exactly when the second type of foliation is not used, i.e., when $b_1(R(f)) = g$.

Remark 19 The conclusion of Theorem 13 also holds, with suitable adjustments, for simple Morse functions on connected closed surfaces of genus g that are non-orientable or have h boundary components. Indeed, [3] gives for these manifolds the following exact bounds:

$0 \le b_1(R(f)) \le g,$	[3, Lemma A]: orientable
$0 \le b_1(R(f)) \le \left[\frac{g}{2}\right],$	[3, Lemma C]: non-orientable
$g \le b_1(R(f)) \le 2g + h - 1,$	[3, Lemma B]: boundary, orientable
$0 \le b_1(R(f)) \le g + h - 1,$	[3, Lemma D]: boundary, non-orientable

with all intermediate values being realizable. However, in all these cases the upper bound coincides with $b_1'(M)$, which is, in particular, consistent with (3). For non-simple Morse functions, the lower bound in the third case can be relaxed along the lines of Fig. 5.

6 Further directions

In this paper, we have generalized the result [3, Lemma A], originally stated for simple Morse functions on a closed orientable surface, to arbitrary smooth functions on closed orientable manifolds of arbitrary dimension.

Currently we work on generalizing our main result (3), and in some form Theorem 13, to a wider class of "nice" topological spaces, including non-orientable or non-closed manifolds and manifolds with boundary. Remark 19 shows that our results hold for all maniflods considered in [3].

However, the topology of the Reeb graph on, for instance, non-closed manifolds can be rather unusual: the Reeb graph of $\mathbb{R} \setminus 0$ with the function $f(x, y) = x$ is path-connected but not arc-connected. What is more, the "obvious" bounds (1), and therefore (3), contrary to a popular belief [5, §VI.4, p. 141] do not hold for some pathological spaces; see Fig. 7.

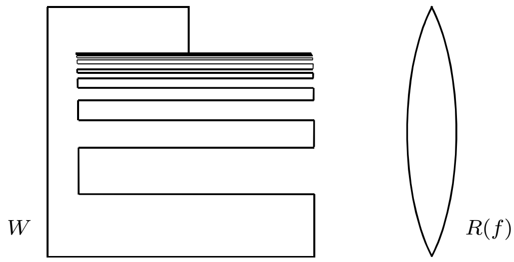

Fig. 7 Left: The Warsaw circle W, obtained by closing the topologist's sine curve, with a height function f. Right: The Reeb graph $R(f)$. The space W is simply connected, $b_1(W) = 0$, while the Reeb graph $R(f)$ is a circle, $b_1(R(f)) = 1$, so that the bounds (1) and (3) do not hold for W. Note that the space W is rather "good", e.g., it is arc-connected.

Note that the results of [3] for surfaces with boundary do not generalize to manifolds of greater dimension in terms of number of holes, as stated in [3, Lemms B, D]. Indeed, the following example shows that a 3-manifold with one hole can have functions with arbitrary $b_1(R(f))$:

Example 20 Consider a three-dimensional manifold M with a height function f. Remove from M the interior of a small two-torus lying flat like a donut lies on the table. This will result in an additional loop in the Reeb graph: the central hole of the torus will be a chord in $R(f)$. If, however, the removed torus is located vertically, like in Fig. 1, then $b_1(R(f))$ does not increase.

Similarly, removing the interior of a surface M_g^2 of genus g, some of its handles being located vertifally and some horizontally, increases $b_1(R(f))$ by any value from zero to g.

The observation from Example 20 can be formulated as follows:

Proposition 21 *Let M, $\dim M \geq 3$, be a smooth manifold, $f : M \to \mathbb{R}$ a smooth non-constant function, and $m \geq b_1(R(f))$ an integer. Then a subset can be removed from M resulting in a manifold M' with one connected component of boundary, such that $b_1(R(f|_{M'})) = m$.*

Proposition 22 *Let M, $\dim M \geq 3$, be a smooth manifold and m an integer. Then a subset can be removed from M resulting in a manifold M' with one connected component of boundary, such that for any integer $0 \leq k \leq m$, there exists a simple Morse function $f : M' \to \mathbb{R}$ with $b_1(R(f)) = k$.*

What is more, the effect of removing the interior of a non-small sufrace from a three-manifold M can depend on the way it is embedded in M: a solid torus embedded in a three-torus T^3 in a contractible way can increase the number of loops in the Reeb graph, but a solid torus embedded in a non-contractible way cannot.

Therefore, the study of the number of loops in the Reeb graph of a function on a manifold M with boundary, or, which is probably the same, the study of $b_1'(M)$, in terms of characteristics of the boundary would involve considering the topology of the boundary in terms of its homology or its fundamental group, as well as considering the way it is embedded in M, and not only the number of connected components of the boundary as in [3].

Conflict of Interest

The author declares that she has no conflict of interest.

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