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Noncompact Leaves of Foliations of Morse Forms

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ABSTRACT. In this paper foliations determined by Morse forms on compact manifolds are considered. An inequality involving the number of connected components of the set formed by noncompact leaves, the number of homologically independent compact leaves, and the number of singular points of the corresponding Morse form is obtained.

KEY WORDS: Morse forms, noncompact leaves of foliations, two-dimensional manifolds.

Consider a Morse form ω that determines a foliation with singularities $\mathcal{F}_\omega$ on a compact manifold M^n. For compact foliations, the author obtained (see [1]) an estimate for the number of homologically independent compact leaves in terms of the number of singular points of the corresponding Morse form. Here we extend this inequality to noncompact foliations.

Further, we obtain an estimate for the number s of connected components of the set formed by noncompact leaves in terms of some characteristics of the Morse form. In [2] Arnoux and Levitt obtained the following estimate of s in terms of characteristics of the manifold M:

$$s \le \frac{1}{2}\beta_1(M^n). \tag{1}$$

In the present paper we show that these two estimates coincide for two-dimensional manifolds, and that they are independent in the general case.

§1. Basic definitions

Consider a smooth compact connected oriented manifold M of dimension n with a closed 1-form Ω having only Morse singularities; below these forms will be called *Morse forms*. By $\operatorname{Sing}\omega$ denote the set of the singular points of ω. A closed form ω determines a foliation $\mathcal{F}$ of codimension 1 on the set $M \setminus \operatorname{Sing}\omega$.

Let us denote a foliation with singularities $\mathcal{F}_\omega$ on M as follows.

Suppose that the foliation $\mathcal{F}$ is locally determined in the neighborhood of a singular point p, $p \in \operatorname{Sing}\omega$, by the level surfaces of functions f_p such that $f_p(p) = 0$. It is clear that $f_p^{-1}(0)\setminus p \subset \bigcup \gamma_i$, where $\gamma_i \in \mathcal{F}$.

A leaf $\gamma \in \mathcal{F}$ is called a *nonsingular leaf* of $\mathcal{F}_\omega$ if $\gamma \cap f_p^{-1}(0) = \varnothing$ for any $p \in \operatorname{Sing}\omega$. Put

$$F_p = p \cup \{\gamma \in \mathcal{F} \mid \gamma \cap f_p^{-1}(0) \ne \varnothing\} \qquad \text{and} \qquad F = \bigcup_{p\in \operatorname{Sing}\omega} F_p.$$

A *singular leaf* γ_0 of $\mathcal{F}_\omega$ is a connected component of F. It is clear that the number of singular points is finite for Morse forms.

Let us assign the homology class $[\gamma]$ to each nonsingular compact leaf $\gamma \in \mathcal{F}_\omega$. Then the image of the set of nonsingular compact leaves generates a subgroup in $H_{n-1}(M)$. We denote this subgroup by H_ω.

Let p be a singular point of a Morse form ω, and let $x^1, \dots, x^n$ be coordinates in the neighborhood of p such that

$$\omega = \sum_{i=1}^{\lambda} x^i\,dx^i - \sum_{i=\lambda+1}^{n} x^i\,dx^i.$$

Then the *index* $\operatorname{ind} p$ of the singular point p is the number $\min\{\lambda, n-\lambda\}$. By Ω_i denote the set of singular points of index i.

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§2. The graph of a foliation

Consider the set $U \subset M$ of all nonsingular compact leaves of $\mathcal{F}_\omega$. By virtue of Lemma 2.1 from [3], for any nonsingular compact leaf γ there exists a neighborhood consisting of leaves homeomorphic to γ. Let us denote by $\mathcal{O}(\gamma)$ the maximal neighborhood with this property. Then $U = \bigcup_\gamma \mathcal{O}(\gamma)$. Note that $\mathcal{O}(\gamma) \simeq \gamma \times \mathbb{R}$ whenever $\operatorname{Sing}\omega \neq \varnothing$.

If the leaves γ and γ' are different, then the sets $\mathcal{O}(\gamma)$ and $\mathcal{O}(\gamma')$ are either disjoint or coincide. Thus there are at most four cylinders $\mathcal{O}(\gamma)$ attached to a singular point p, i.e., $p \in \bigcup_{i=1}^k \overline{\mathcal{O}(\gamma_i)}$, where $k \leq 4$, and thus the number of different sets $\mathcal{O}(\gamma)$ is finite. Consequently, $U = \bigcup_{i=1}^N \mathcal{O}(\gamma_i)$.

The set $M \setminus U$ consists of singular compact and noncompact leaves of ω. The number of singular compact leaves is finite; further, it was shown in [2] that the number of connected components of the set formed by noncompact leaves is finite as well. Thus the set $M \setminus U$ consists of a finite number of connected components W_k.

Let us consider the closure of U:

$$\overline{U} = \bigcup_{i=1}^{N} V_i, \qquad \text{where} \quad V_i = \overline{\mathcal{O}(\gamma_i)}.$$

By Lemma 2.2 from [3], each connected component of the boundary ∂V_i lies in some singular leaf γ_0. Since the singular leaves lie in $M \setminus U$, we see that the set W_k such that $W_k \supset \gamma_0$ is assigned to each singular leaf γ_0. Consequently, the sets W_k correspond to the connected components of the boundary ∂V_i, and $\partial V_i \cap W_k \neq \varnothing$.

Let us assign a finite graph Γ to M by identifying the sets V_i with its edges and the sets W_k with its vertices. An edge V_i is incident to a vertex W_k if $\partial V_i \cap W_k \neq \varnothing$. The graph Γ is called the *associated graph* of the foliation $\mathcal{F}_\omega$.

There are two types of vertices W_k of Γ. If the singular leaf $\gamma_0 \subset W_k$ is compact, then $\gamma_0 = W_k$, because Lemma 2.5 from [3] implies that a sufficiently small neighborhood of a compact singular leaf consists of nonsingular compact leaves. We say that the vertices $W_k = \gamma_0$ are *vertices of the first type* and we denote them by γ_0; the corresponding singular leaves are compact. The vertices $W_k \neq \gamma_0$ are said to be *vertices of the second type*; the corresponding singular leaves are noncompact.

Consider vertices of the first type: the singular leaf γ_0 is compact. If $\gamma_0 \cap \Omega_0 \neq \varnothing$, then the degree of the corresponding vertex is equal to 1. If $|\gamma_0 \cap \Omega_1| = m > 0$, then Theorem 3.1 from [3] implies the estimate $\deg \gamma_0 \leq m + 2$ for the degree of the corresponding vertex. If $\gamma_0 \cap (\Omega_0 \cup \Omega_1) = \varnothing$, then this vertex has degree 2.

Consider vertices of the second type: the singular leaves $\gamma_0 \subset W_k$ (along which the sets V_i are glued) are noncompact. Obviously, we have $\gamma_0 \cap \Omega_0 = \varnothing$. If $|\gamma_0 \cap \Omega_1| = m > 0$, then Theorem 3.1 from [3] implies that the number of the sets V_i glued to γ_0 does not exceed m. If $\gamma_0 \cap \Omega_1 = \varnothing$, then it is easy to see that $\gamma_0 \cap \partial V_i = \varnothing$.

So, taking the sum over all the singular leaves lying in W_k, we see that if $|W_k \cap \Omega_1| = m$, then the degree of W_k is at most m. Note that the relation $m = 0$ holds only if the foliation contains no compact leaves.

The graph Γ is connected.

§3. The main theorem

Let us denote by s the number of connected components of the set formed by noncompact leaves.

Theorem 1. *The following inequality holds*:

$$\operatorname{rk} H_\omega + s \leq \frac{1}{2}(|\Omega_1| - |\Omega_0|) + 1.$$

Proof. Consider the graph Γ of a foliation $\mathcal{F}_\omega$. By $m(\Gamma)$ denote the cyclic rank of the graph Γ, by Q denote the number of its edges, and by $P^{(j)}$ the number of vertices of the jth type, $j = 1, 2$. By Theorem 4.5 from [4], for connected graphs we have the following relation:

$$m(\Gamma) = Q - (P^{(1)} + P^{(2)}) + 1.$$

Let $k_i^{(j)}$ be the number of vertices of the jth type of degree i. Then

$$P^{(j)} = \sum_{i>0} k_i^{(j)},$$

and Theorem 2.1 from [4] implies that

$$2Q = \sum_{i>0} i(k_i^{(1)} + k_i^{(2)}).$$

Note that $s = P^{(2)}$. So we have the relation

$$2m(\Gamma) + 2s = -k_1^{(1)} + \sum_{i>1}(i-2)k_i^{(1)} + \sum_{i>0} ik_i^{(2)} + 2.$$

It is clear that $k_1^{(1)} = |\Omega_0|$. By Theorem 3.1 from [3], the second summand is less than or equal to the number of singular points of index 1 belonging to compact leaves. As shown above, that $\deg W_k \le |W_k \cap \Omega_1|$; consequently, the third summand is less than or equal to the number of singular points of index 1 belonging to noncompact leaves. Thus we obtain the inequality

$$2m(\Gamma) + 2s \le |\Omega_1| - |\Omega_0| + 2.$$

Let us examine the cyclic rank of the graph Γ.

Lemma 1. *The inequality* $\operatorname{rk} H_\omega \le m(\Gamma)$ *holds.*

Proof. Each vertex of the associated graph Γ imposes a linear equation on the homology classes of the leaves γ_i, where $\gamma_i \in V_i$, and so the graph Γ determines a system of P linear equations in Q variables, where P is the number of vertices of the graph and Q is the number of its edges. The rank $\operatorname{rk} H_\omega$ does not exceed the rank of the space of solutions of this system. The matrix of this system is the incidence $P \times Q$ matrix of the connected graph Γ. Here the columns corresponding to the loops are zero columns, because a loop is incident to the corresponding vertex twice with opposite signs. By virtue of Theorem 13.6 from [4], the rank of the incidence matrix is equal to $P - 1$. Consequently, $\operatorname{rk} H_\omega \le Q - P + 1 = m(\Gamma)$. The lemma is proved. □

Now Lemma 1 yields the inequality

$$2 \operatorname{rk} H_\omega + 2s \le |\Omega_1| - |\Omega_0| + 2.$$

This completes the proof of the theorem. □

This theorem implies the following upper bound for the number s of the connected components of the set formed by noncompact leaves:

$$s \le \frac{1}{2}(|\Omega_1| - |\Omega_0|) + 1. \tag{2}$$

Let us show that estimates (1) and (2) are independent.

Consider a winding of the torus T^n determined by a nonsingular 1-form with constant coefficients. Since $|\Omega_1| = |\Omega_0| = 0$, it follows from (2) that $s \le 1$. Further, since $\beta_1(T^n) = n$, we see that $s \le n/2$, and for $n > 2$ estimate (2) is sharper.

Now consider a manifold M^n such that $\beta_1(M^n) = 1$, and a foliation determined by a height function. Then $|\Omega_0| = 2$, $|\Omega_1| \ge 2$, and $s \le 1$ because of estimate (2). On the other hand, it follows from (1) that $s \le 1/2$. Thus estimate (1) is sharper in this case.

The following statement shows that the two estimates are equivalent in the two-dimensional case.

Corollary 1. *If M_g^2 is a two-dimensional manifold, then the following inequality holds:*

$$\operatorname{rk} H_\omega + s \leq g.$$

Proof. For two-dimensional manifolds M_g^2, we have $|\Omega_1| - |\Omega_0| = 2g - 2$. An application of Theorem 1 completes the proof. □

The ratio of the numbers of singular points of indices 0 and 1 sometimes enables one to obtain certain conclusions concerning the foliation.

Corollary 2. *If $|\Omega_0| > |\Omega_1|$, then the foliation is compact and all its leaves are homologous to zero.*

Proof. If $|\Omega_0| > |\Omega_1|$, then Theorem 1 implies the relations $\operatorname{rk} H_\omega = s = 0$. □

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