On compact leaves of a Morse form foliation

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Abstract. On a compact oriented manifold without boundary, we consider a closed 1-form with singularities of Morse type, called Morse form. We give criteria for the foliation defined by this form to have a compact leaf, to have $k$ homologically independent compact leaves, and to have no minimal components.

1. Introduction and the results

Consider a compact oriented connected smooth $n$-dimensional manifold $M$ without boundary. On $M$, consider a smooth differential 1-form $\omega$ that is closed, i.e., $d\omega = 0$. By the Poincaré lemma, it is locally the differential of a function: $\omega = df$.

In this paper, we assume $f$ to be a Morse function; then $\omega$ is called a Morse form. By Morse functions we mean smooth functions with non-degenerate singularities. They are generic (typical) smooth functions: their set is open and dense in the space of smooth functions [7]. Likewise, Morse forms are generic (typical) closed 1-forms: their set is open and dense in the space of all closed 1-forms on $M$.

Let $\omega$ be a Morse form on $M$. The set of its singularities $\text{Sing} \omega = \{ x \in M \mid \omega_x = 0 \}$ is finite. On $M \setminus \text{Sing} \omega$ the form $\omega$ defines a foliation $\mathcal{F}_\omega$ constructed as follows: For any $x \in M \setminus \text{Sing} \omega$, the equation $\{ \omega_x(\xi) = 0 \}$ defines a distribution of the tangent bundle $T_x M$. Since $\omega$ is closed, this distribution is integrable; its integral surfaces are leaves of $\mathcal{F}_\omega$.

A foliation is a way of slicing the manifold into disjoint submanifolds (called leaves) of lower dimension, in our case the dimension $n - 1$. This notion is widely

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used in physics. For example, the phase space of a mechanical system is foliated by its energy levels. Foliations of space-time into three-dimensional space-like hypersurfaces have been found to completely characterize the topology of space-time, the singularities describing the topological structure of the gravitational singularities [10].

A foliation $\mathcal{F}_\omega$ has three types of leaves: compact, non-compact compactifiable and non-compact non-compactifiable. If a leaf $\gamma$ is compactified by $\text{Sing}_\omega$, i.e., $\gamma \cup \text{Sing}_\omega$ is compact, then it is called compactifiable, otherwise it is called non-compactifiable. In particular, compact leaves are compactifiable. A foliation is called compactifiable if it has only compactifiable leaves, i.e., if it has no minimal components (areas covered by non-compactifiable leaves).

Existence of compact leaves and existence of non-compactifiable leaves in a given foliation are classical problems of the foliation theory. We consider both these problems for a Morse form foliation.

Denote by $H_{\omega} \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$, and by $\mathcal{H} \subseteq H_{\omega}$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $\cdot$ is the cycle intersection and $[\omega] : H_1(M) \to \mathbb{R}$ the integration map. We denote $\text{rk}_\omega \equiv \text{rk}_Q \text{Im}[\omega]$.

Melnikova [9] has shown that on a two-dimensional manifold, a foliation $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} \geq \text{rk}\omega - 1$. We generalize this fact to arbitrary dimension and give a stronger formulation: $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} = \text{rk}\omega$ (Theorem 8). In Theorem 8 we also show that $\text{rk}\mathcal{H} \leq \text{rk}\omega$, but $\text{rk}\mathcal{H} \neq \text{rk}\omega - 1$.

Farber et al. [2], [3] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form. We show that this condition is not a criterion. Then we generalize it to arbitrary (not necessarily transitive) Morse forms and improve it to a criterion.

For this, we introduce the notion of collinearity of forms: we call a (not necessarily Morse) smooth closed 1-form $\alpha$ collinear with $\omega$ if $\alpha \wedge \omega = 0$; foliations of collinear forms share entire leaves (Proposition 14). We give a criterion for existence of compact leaves: $\mathcal{F}_\omega$ has a compact leaf iff there exists a form $\alpha \neq 0$ collinear with $\omega$ such that $[\alpha] \in H^1(M, \mathbb{Z})$ (Theorem 16); what is more, $\mathcal{F}_\omega$ has $k$ homologically independent compact leaves iff there exist $k$ cohomologically independent such forms (Theorem 18).

Finally, we give a condition for compactifiability of $\mathcal{F}_\omega$ in terms of existence of a sufficient number of cohomologically independent forms collinear with $\omega$ (Theorem 18).

The paper is organized as follows. In Section 2, we introduce necessary definitions and facts about Morse form foliations. In Section 3, we prove a criterion
for a Morse form foliation to be compactifiable. Finally, in Section 4 we introduce a notion of collinearity of 1-forms and use it to give criteria for a foliation to have a compact leaf or \( k \) homologically independent compact leaves.

2. Definitions and useful facts

Recall that \( M \) is a compact oriented connected smooth \( n \)-dimensional manifold without boundary.

2.1. Poincaré duality map. We call an injection \( D : H_{n-1}(M) \hookrightarrow H_1(M) \) a Poincaré duality map if there exists a basis \( z_i \in H_{n-1}(M) \) such that

\[
z_i \cdot Dz_j = \delta_{ij},
\]

where \( \cdot \) is the intersection form. For any basis \( z_i \in H_{n-1}(M) \), there exists a Poincaré duality map satisfying (1). Obviously, if a subgroup \( G \subseteq H_{n-1}(M) \) is a direct summand in \( H_{n-1}(M) \), i.e. \( H_{n-1}(M) = G \oplus G' \) for some \( G' \), then for any basis \( z_i \in G \) there exists a Poincaré duality map satisfying (1).

Note that for any subgroup \( G \subseteq H_{n-1}(M) \) we have an isomorphism \( DG \cong G \); in particular, \( \text{rk} DG = \text{rk} G \).

2.2. A Morse form foliation. Recall that for a Morse form \( \omega \), the set \( \text{Sing} \omega \) is finite since the singularities are isolated and \( M \) is compact; on \( M \setminus \text{Sing} \omega \) the form defines a foliation \( \mathcal{F}_\omega \). The number of its non-compact compactifiable leaves is finite, since each singularity can compactify no more than four leaves. The union of all non-compactifiable leaves is open and has a finite number \( m(\omega) \) of connected components [1] called minimal components; we call compactifiable a foliation that has no minimal components.

For a compact leaf \( \gamma \) there exists an open neighborhood consisting solely of compact leaves: indeed, integrating \( \omega \) gives near \( \gamma \) a function \( f \) with \( df = \omega \). Hence, the union of all compact leaves is open. Denote by \( H_\omega \subseteq H_{n-1}(M) \) a group generated by all compact leaves of \( \mathcal{F}_\omega \). A Morse form foliation defines the following decomposition [4]:

\[
H_1(M) = DH_\omega \oplus i_*H_1(\Delta),
\]

where \( \Delta \) is the union of all non-compact leaves and singularities, \( i : \Delta \hookrightarrow M \), and \( D : H_{n-1}(M) \to H_1(M) \) is a Poincaré duality map.
The value \( c(\omega) = \text{rk} H_\omega \) is the number of homologically independent compact leaves, i.e. \( H_\omega \) has a basis of homology classes of compact leaves, \( H_\omega = \langle [\gamma_1], \ldots, [\gamma_{c(\omega)}] \rangle \) [4]. For a compactifiable foliation, (2) gives

\[
c(\omega) \geq \text{rk} \omega, \tag{3}
\]

where \( \text{rk} \omega = \text{rk} \text{im}[\omega] \), with \( [\omega] : H_1(M) \to \mathbb{R} \) being the integration map. Obviously,

\[
\text{rk} \omega + \text{rk} \ker[\omega] = b_1(M), \tag{4}
\]

the first Betti number.

### 2.3. Non-commutative Betti number

Arnoux and Levitt [1] denoted by \( b'_1(M) \) the non-commutative Betti number — the maximal rank (number of free generators) of a free quotient group of \( \pi_1(M) \); note that \( b'_1(M) \leq b_1(M) \) [8].

**Example 1.** For an \( n \)-dimensional torus we have \( b'_1(T^n) = 1 \); for the connected sum \( \# \) of direct products \( S^1 \times S^n, n > 1 \), we have \( b'_1(\#_{i=1}^p (S^1 \times S^n)) = p \); for a genus \( g \) two-dimensional surface we have \( b'_1(M^2_g) = g \) [5].

The topology of the foliation is connected with \( b'_1(M) \) [5]:

\[
c(\omega) + m(\omega) \leq b'_1(M), \tag{5}
\]

where \( c(\omega) \) is the number of homologically independent compact leaves and \( m(\omega) \) the number of minimal components.

Denote by \( h(M) \) the maximum number of cohomologically independent cochains \( u_i \in H^1(M, \mathbb{Z}) \) such that the cup-product \( u_i \smile u_j = 0 \) [4]. Then \( c(\omega) \leq h(M) \) [6, Theorem 3.2] and for some Morse form \( \omega \) on \( M \) [5, Theorem 8] it holds

\[
c(\omega) = b'_1(M), \tag{6}
\]

which gives

\[
b'_1(M) \leq h(M). \tag{7}
\]

### 3. Conditions for compactifiability

Denote by \( H \subseteq H_\omega \) a subgroup of all \( z \in H_\omega \) such that \( z \cdot \ker[\omega] = 0 \), where \( H_\omega \subseteq H_{n-1}(M) \) is the subgroup generated by all compact leaves of \( \mathcal{F}_\omega \) and \( \cdot \) is the cycle intersection.

**Lemma 2.** It holds:
(i) $H_\omega$ is a direct summand in $H_{n-1}(M)$;
(ii) $\mathcal{H}$ is a direct summand in $H_\omega$;
(iii) $\mathcal{H}$ is a direct summand in $H_{n-1}(M)$.

**Proof.** It is easy to show that a subgroup of a finitely-generated free abelian group is a direct summand iff its quotient is torsion-free.

(i) Let us show that the quotient $H_{n-1}(M)/H_\omega$ is torsion-free. It has been shown in [4] that there exist compact leaves $\gamma_1, \ldots, \gamma_c(\omega) \in \mathcal{F}_\omega$ and closed curves $\alpha_1, \ldots, \alpha_c(\omega) \subset M$ such that $[\gamma_i]$ form a basis of $H_\omega$ and $[\gamma_i] \cdot [\alpha_j] = \delta_{ij}$. Suppose there exists $0 \neq z = z_0 + H_\omega \in H_{n-1}(M)/H_\omega$ such that $kz = 0$ for some $0 \neq k \in \mathbb{Z}$, i.e., $z_0 \notin H_\omega$ but $kz_0 \in H_\omega$. Then $kz_0 = \sum n_i[\gamma_i]$ and $kz_0 \cdot [\alpha_j] = n_j$. Consider $z_1 = \sum \frac{1}{k_i} [\gamma_i] \in H_\omega$, then $kz_1 = kz_0$. Since $H_\omega \subseteq H_{n-1}(M)$ is torsion-free, we obtain $z_0 = z_1 \in H_\omega$; a contradiction.

(ii) Let us show now that the quotient $H_\omega/\mathcal{H}$ is torsion-free. Similarly, suppose $z \notin \mathcal{H}$, i.e., $z \cdot \ker[\omega] \neq 0$, then $kz \cdot \ker[\omega] \neq 0$ and thus $kz \notin \mathcal{H}$.

(iii) follows from (i) and (ii). \(\square\)

Recall that $D : H_{n-1}(M) \to H_1(M)$ is a Poincaré duality map defined by the cycle intersection. By Lemma 2, for a basis $z_i \in \mathcal{H}$ there exists a Poincaré duality map that satisfies (1).

**Lemma 3.** Let $z_i$ be a basis of $\mathcal{H} \subset H_{n-1}(M)$ and $D$ a corresponding Poincaré duality map. Then the integrals $\int_{Dz_i} \omega$ are independent over $\mathbb{Q}$.

Indeed, suppose $\sum n_i \int_{Dz_i} \omega = 0$, i.e., $z = \sum n_i Dz_i \in \ker[\omega]$. Then $n_i = z \cdot z_i = 0$.

**Proposition 4.** If $\text{rk } \mathcal{H} \geq \text{rk } \omega - 1$ then $\mathcal{F}_\omega$ is compactifiable.

In fact we will show below that $\text{rk } \mathcal{H} \neq \text{rk } \omega - 1$, so the above inequality is equivalent to $\text{rk } \mathcal{H} = \text{rk } \omega$.

**Proof.** Consider a basis $z_i \in \mathcal{H}$ and a corresponding Poincaré duality map $D$. Denote $L_{\mathcal{H}} = \langle \int_{Dz_i} \omega \rangle$, a linear space over $\mathbb{Q}$; by Lemma 3, dim $L_{\mathcal{H}} = \text{rk } \mathcal{H}$.

Suppose that there exists a minimal component $U$. Then $\text{rk } \omega|_U \geq 2$, i.e., there exist two cycles $s, u \in i_*H_1(U)$, where $i : U \hookrightarrow M$, with independent periods [8]. Denote $L_U = \langle \int_s \omega, \int_u \omega \rangle$, a linear space over $\mathbb{Q}$; dim $L_U = 2$.

Let us show that $L_{\mathcal{H}} \cap L_U = 0$. Consider $z = n_s s + n_u u$ such that $\int_z \omega \in L_{\mathcal{H}}$, i.e. $\int_z \omega = \sum n_i \int_{Dz_i} \omega$. Thus $z - \sum n_i Dz_i \in \ker[\omega]$. By definition, $\mathcal{H} \cdot \ker[\omega] = 0$, so $z_j : (z - \sum n_i Dz_i) = 0$. Since $z_j$ are generated by compact leaves while $z \in i_*H_1(U)$; we have $z_j \cdot z = 0$. This gives all $n_j = 0$ and thus $\int_z \omega = 0$.

We have $\text{rk } \omega \geq \text{dim } (L_{\mathcal{H}} \cup L_U) = \text{rk } \mathcal{H} + 2$; a contradiction. \(\square\)
The following condition in terms of compact leaves is geometrically more visual than Proposition 4:

**Corollary 5.** Let $F_\omega$ have $\text{rk}_\omega - 1$ homologically independent compact leaves $\gamma_i$ such that $[\gamma_i] \cdot \ker[\omega] = 0$. Then $F_\omega$ is compactifiable, and there exists another compact leaf $\gamma$ homologically independent from all $\gamma_i$.

**Proof.** By Proposition 4, the foliation is compactifiable. Then (3) gives $c(\omega) \geq \text{rk}\omega$, so there exists a compact leaf $\gamma$ such that $[\gamma] \notin ([\gamma_i])$. □

Corollary 5 is not a criterion:

**Counterexample 6.** On a two-dimensional genus 4 surface $M_4^2$ represented as a connected sum of four tori $T^2$, consider a compactifiable foliation such that $\int z_1 \omega = \int z_2 \omega = 1$ and $\int z_3 \omega = \int z_4 \omega = \sqrt{2}$, so that $\text{rk}\omega = 2$ and $c(\omega) = 4$; see Figure 1. Then $(z_1 - z_2), (z_3 - z_4) \in \ker[\omega]$, but for any homologically non-trivial compact leaf $\gamma \in F_\omega$ we have either $[\gamma] \cdot (z_1 - z_2) \neq 0$ or $[\gamma] \cdot (z_3 - z_4) \neq 0$, so there are no $\text{rk}\omega - 1 = 1$ homologically independent leaves such that $[\gamma] \cdot \ker[\omega] = 0$. Note that still $\text{rk}\mathcal{H} = 2$, cf. Proposition 4.

\[
\begin{array}{c}
\text{z}_1 \\
\text{z}_2 \\
\text{z}_3 \\
\text{z}_4
\end{array}
\begin{array}{c}
1 \\
1 \\
\sqrt{2} \\
\sqrt{2}
\end{array}
\]

*Figure 1.* A foliation on a connected sum $M_4^2 = \#_4 T^2$.

However, with an additional condition the converse to Corollary 5 is true:

**Proposition 7.** If $F_\omega$ is compactifiable and $\text{rk}\omega = c(\omega)$, then there exist $\text{rk}\omega$ homologically independent compact leaves $\gamma_i \in F_\omega$ such that $[\gamma_i] \cdot \ker[\omega] = 0$. 

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PROOF. Consider a basis $[\gamma_i] \in H_\omega$. For a compactifiable foliation, $\Delta$ mentioned in (2) is the union of a finite number of compactifiable leaves and singularities, so $i_*H_1(\Delta) \subseteq \ker[\omega]$ and $\text{rk}\omega$ is determined by $D[\gamma_i]$. Since $\text{rk}(D[\gamma_i]) = \text{rk}H_\omega = c(\omega) = \text{rk}\omega$, all corresponding integrals are rationally independent, so $\ker[\omega] = i_*H_1(\Delta)$. Then $[\gamma_i] \cdot \ker[\omega] = 0$ since $\gamma_i \cap \Delta = \emptyset$. □

In the rest of this section we will study $\text{rk}\mathcal{H}$. By (5), $\text{rk}\mathcal{H} \leq b'_1(M)$. The following properties of $\text{rk}\mathcal{H}$ are connected with $\text{rk}\omega$:

**Theorem 8.** It holds:

(i) $\text{rk}\mathcal{H} \leq \text{rk}\omega$.

(ii) $\text{rk}\mathcal{H} \neq \text{rk}\omega - 1$.

(iii) $F_\omega$ is compactifiable iff $\text{rk}\mathcal{H} = \text{rk}\omega$.

**Proof.** (i) follows from Lemma 3.

(ii) follows from Proposition 4 and (iii).

(iii) If $\text{rk}\mathcal{H} = \text{rk}\omega$ then $F_\omega$ is compactifiable by Proposition 4.

Let now $F_\omega$ be compactifiable. By Lemma 2 there exists a Poincaré duality map $D$ that satisfies (1) for a basis $[\gamma_i]$ of $H_\omega$. Consider $\varphi : H_\omega \to \mathbb{R}$, $\varphi(z) = \int_{Dz} \omega$. Since $\Delta$ in (2) consists of a finite number of compactifiable leaves and singularities, we have $i_*H_1(\Delta) \subseteq \ker[\omega]$; in particular, $\text{rk}\omega = \text{rk}\text{im}\varphi$.

Recall that $\mathcal{H} = \{z \in H_\omega \mid z \cdot \ker[\omega] = 0\}$. Let $u = u_1 + u_2 \in H_1(M)$, $u_1 \in DH_\omega$, $u_2 \in i_*H_1(\Delta)$ according to (2). Since $H_\omega \cdot i_*H_1(\Delta) = 0$, we have $z \cdot u = z \cdot u_1$. For a compactifiable foliation, $u_2 \in \ker[\omega]$, so $u \in \ker[\omega]$ if $u_1 \in \ker[\omega]$. Thus the above definition can be rewritten as $\mathcal{H} = \{z \in H_\omega \mid z \cdot DH_0 = 0\}$, where $H_0 = \ker\varphi$; in other words, $\mathcal{H}$ is the set of all $z = \sum n_i[\gamma_i]$ such that for all $z_k = \sum m_{ki}[\gamma_i]$ that generate $H_0 \subseteq H_\omega$ it holds $z \cdot Dz_k = 0$, i.e., $\sum n_i m_{ki} = 0$.

The latter linear system implies $\text{rk}\mathcal{H} = \text{rk}H_\omega - \text{rk}H_0$. Since $\text{rk}H_\omega = c(\omega)$ and $\text{rk}H_0 = \text{rk}\ker\varphi = \text{rk}H_\omega - \text{rk}\text{im}\varphi = c(\omega) - \text{rk}\omega$, we obtain $\text{rk}\mathcal{H} = \text{rk}\omega$. □

Let us consider some special cases.

**Corollary 9.** Let $\ker[\omega] = 0$. Then $F_\omega$ is compactifiable iff $c(\omega) = b_1(M)$, the first Betti number. In this case the cup-product $\sim : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is trivial.

**Proof.** Since $\ker[\omega] = 0$, we have $\text{rk}\omega = b_1(M)$ and $\mathcal{H} = H_\omega$. Then the condition $\text{rk}\mathcal{H} = \text{rk}\omega$ from Theorem 8 (iii) is equivalent to $c(\omega) = b_1(M)$ and thus $H_\omega = H_{n-1}(M)$. Then $H^1(M, \mathbb{Z}) = \langle z \rangle$, where $z$ are cocycles dual to $[\gamma_i]$, a basis of $H_\omega$, and $\gamma_i \cap \gamma_j = \emptyset$ implies $z_i \sim z_j = 0$. □
So if \( \ker[\omega] = 0 \) and \( \sim \not\equiv 0 \), then \( F_\omega \) has a minimal component. If, however, \( \sim \equiv 0 \), then both cases are possible. Indeed, on the one hand, in any cohomology class \( [\omega] \), \( \text{rk}\, \omega > 1 \), there exists a Morse form with minimal foliation \([1]\). On the other hand, the foliation can be compactifiable:

**Example 10.** Consider a connected sum \( M = \bigoplus_{i=1}^p (S_1 \times S^n) \), \( n > 1 \); see Figure 2. Then \( b_1(M) = b'_1(M) = p \) (Example 1), which by (7) gives \( \sim \equiv 0 \). Consider \( \omega \) given on each \((S_1 \times S^n)_i\) by \( \omega_i = \alpha_i dt \), where \( t \) is a coordinate on \( S_1 \) and all \( \alpha_i \in \mathbb{R} \) are independent over \( \mathbb{Q} \) so that \( \text{rk}\, \omega = p \). Obviously, \( F_\omega \) is compactifiable (its compact leaves are spheres \( S^n \)).

![Figure 2. A foliation on a connected sum \((S_1 \times S^n) \bigoplus (S_1 \times S^n)\).](image)

**Corollary 11.** For a two-dimensional genus \( g \) surface \( M_g^2 \) it holds

\[
\text{rk}\, H \leq \text{rk}\, \omega \leq 2g - c(\omega) \leq 2g - \text{rk}\, H.
\]

(8)

If \( \text{rk}\, H = g \), then \( F_\omega \) is compactifiable.

**Proof.** The lower bound is by Theorem 8 (i). Since leaves are one-dimensional, \( H \subseteq H_\omega \subseteq \ker[\omega] \) and \( \text{rk}\, \ker[\omega] = 2g - \text{rk}\, \omega \) gives the upper bound. If \( \text{rk}\, H = g \) then (8) implies \( \text{rk}\, \omega = g \) and \( F_\omega \) is compactifiable by Theorem 8 (iii). \( \Box \)

4. Criterion for the presence of compact leaves

Farber et al. proved a necessary condition of existence of a compact leaf \( \gamma \) in terms of zero cup-product:

**Proposition 12** ([2, Proposition 9.14],[3, Proposition 3]). For so-called transitive Morse forms, if \( F_\omega \) has a compact leaf with \([\gamma] \neq 0\) then there exists a smooth closed 1-form \( \alpha \), \( 0 \neq [\alpha] \in H^1(M, \mathbb{Z}) \), such that \([\alpha] \sim [\omega] = 0\).
The converse is, however, not true; see Counterexample 17 below. Moreover, no sufficient conditions for existing of a compact leaf can be given in cohomological terms: any cohomology class \([\omega], \text{rk } \omega > 1\), contains a form with minimal foliation [1].

We call 1-forms \(\alpha\) and \(\beta\) collinear if \(\alpha \wedge \beta = 0\). Using the notion of collinearity instead of zero cup-product, we will generalize Proposition 12 to an arbitrary (not necessarily transitive) Morse form and refine it to a criterion. For closed 1-forms the equation \(\alpha \wedge \beta = 0\) implies \([\alpha] \sim [\beta] = 0\) but not vice versa, so collinearity is a stronger condition.

Denote \(\text{Supp } \alpha = M \setminus \text{Sing } \alpha\). If \(\alpha\) is closed, on \(\text{Supp } \alpha\) the integrable distribution \(\{\alpha = 0\}\) defines a foliation \(F_\alpha\).

**Lemma 13.** For closed collinear 1-forms \(\alpha, \beta\), on \(\text{Supp } \beta\) it holds \(\alpha = f(x)\beta\), where \(f(x)\) is constant on leaves of \(F_\beta\). In particular, on \(\text{Supp } \alpha \cap \text{Supp } \beta\) it holds \(F_\alpha = F_\beta\).

**Proof.** On \(\text{Supp } \beta\) there exists a smooth vector field \(\xi_x\) such that \(\beta(\xi_x) \neq 0\). Consider \(f(x) = \frac{\alpha(\xi_x)}{\beta(\xi_x)}\), which is well-defined: for any vector fields \(\xi_x, \eta_x\) we have \(\alpha(\xi_x)\beta(\eta_x) - \alpha(\eta_x)\beta(\xi_x) = (\alpha \wedge \beta)(\xi_x, \eta_x) = 0\). Thus on \(\text{Supp } \beta\) we have \(\alpha = f(x)\beta\).

Since \(\alpha\) and \(\beta\) are closed, \(df \wedge \beta = d\alpha - f d\beta = 0\). Consider a vector field \(\xi\) tangent to the leaves of \(F_\beta\) and \(\eta\) normal to the leaves. Then \(df \wedge (\beta, \eta) = 0\) implies \((df)(\xi) = 0\), i.e. \(f\) is constant on leaves. \(\square\)

**Proposition 14.** Let \(\alpha\) be a smooth closed 1-form collinear with a Morse form \(\omega\); \(\alpha \neq 0\) and \([\alpha] \in H^1(M, \mathbb{Z})\). Then \(\text{Supp } \alpha\) is the union of a non-empty subset of compact leaves of \(F_\omega\) and a subset of compactifiable leaves of \(F_\omega\). These leaves of \(F_\omega\) are leaves of \(F_\alpha\).

**Proof.** All leaves of \(F_\alpha\) are closed. Indeed, since \([\alpha] \in H^1(M, \mathbb{Z})\), it defines a smooth map \(F_{[\alpha]} : M \to S^1\),

\[F_{[\alpha]}(x) = e^{2\pi i \int_{x_0}^x \alpha}.\]

Obviously, \(F_{[\alpha]}\) is constant on leaves of \(F_\alpha\) and the critical set of \(F_{[\alpha]}\) coincides with \(\text{Sing } \alpha\). So on \(\text{Supp } \alpha\) the map is regular and by the implicit function theorem each leaf of \(F_\alpha\) (which is a connected component of a level \(F_{[\alpha]}^{-1}(y), y \in S^1\)) is a closed codimension-one submanifold of \(\text{Supp } \alpha\) (not necessarily closed in \(M\)).

Next, if for a leaf \(\gamma \in F_\omega\) it holds \(\gamma \cap \text{Supp } \alpha \neq \emptyset\) then \(\gamma \subseteq \text{Supp } \alpha\). Indeed, suppose there exists \(x_0 \in \gamma \cap \text{Sing } \alpha\). By Lemma 13, on \(\text{Supp } \omega\) it holds \(\alpha = f(x)\omega\),
where the function $f(x)$ is constant on leaves. Since $x_0 \in \text{Supp}\omega$, we have $f(x_0) = 0$ and so $f|_\gamma = 0$, which gives $\gamma \cap \text{Supp}\alpha = \emptyset$; a contradiction.

Similarly, if for a leaf $\gamma \in F_\alpha$ it holds $\gamma \cap \text{Sing}\alpha \neq \emptyset$ then $\gamma \subseteq \text{Sing}\alpha$. However, since $\text{Sing}\omega$ consists of isolated points, such a leaf $\gamma$ would be a point. This gives $\text{Supp}\alpha \cap \text{Sing}\omega = \emptyset$ and thus $\text{Supp}\alpha \subseteq \text{Supp}\omega$.

Now Lemma 13 implies that all leaves of $F_\alpha$ are leaves of $F_\omega$. Since all leaves of $F_\alpha$ are closed in $\text{Supp}\alpha$, the latter cannot contain any non-compactifiable leaves of $F_\omega$. It cannot consist solely of non-compact compactifiable leaves of $F_\omega$ since their number is finite while $\text{Supp}\alpha$ is open. Thus it must contain compact leaves of $F_\omega$.

**Lemma 15.** In the conditions of Proposition 14, if $[\alpha] \neq 0$ then $F_\alpha$ has a compact leaf with $[\gamma] \neq 0$.

**Proof.** Following the reasoning of [4] it is easy to show that (2) holds for $\alpha$ even though it is not a Morse form. Since its $\Delta$ consists of $\text{Sing}\alpha$ and a finite number of compactifiable leaves, $\text{rk}\alpha$ is determined by $DH\alpha$. However, if $[\gamma] = 0$ for any compact $\gamma \in F_\alpha$ then $H\alpha = 0$ and thus $\text{rk}\alpha = 0$, i.e., $[\alpha] = 0$.

Now we are ready to prove the main result of this section: a criterion for existence of a compact leaf.

**Theorem 16.** The following conditions are equivalent:

(i) $F_\omega$ has a compact leaf $\gamma$;

(ii) There exists a smooth function $f(x) \neq \text{const}$ such that $df$ is collinear with $\omega$;

(iii) There exists a smooth closed 1-form $\alpha \neq 0$, $[\alpha] \in H^1(M, \mathbb{Z})$, collinear with $\omega$.

Moreover, $\gamma$ can be chosen with $[\gamma] \neq 0$ iff $\alpha$ can be chosen with $[\alpha] \neq 0$.

Note that $f$ and $\alpha$ are not required to be of Morse type.

**Proof.** (i) $\Rightarrow$ (ii), (iii): Let $\gamma$ be a compact leaf. Consider a cylindrical neighborhood $O(\gamma) = \gamma \times I$ consisting of diffeomorphic leaves. Let $(x^1, \ldots, x^n)$ be local coordinates in $O(\gamma)$ such that $(x^1, \ldots, x^{n-1})$ are coordinates in $\gamma$ and $x^n$ in $I$. Consider a smooth function $f(x) = f(x^n) \neq \text{const}$ in $O(\gamma)$ and $f(x) = 0$ on $M \setminus O(\gamma)$. Let $x \in O(\gamma)$; consider the leaf $\gamma' \ni x$. Let $\eta_1, \eta_2 \in T_xM$; then $\eta_i = \xi_i + a_i n$, where $\xi_i \in T_x\gamma'$, $a_i \in \mathbb{R}$, and $n \in T_zM \setminus T_x\gamma'$. We obtain $df(\eta_i) = a_i df(n)$ and $\omega(\eta_i) = a_i \omega(n)$. Thus $df \wedge \omega(\eta_1, \eta_2) = 0$, which proves (ii).

Consider now $\alpha = f(x)\omega$; obviously, $\alpha$ is closed and collinear with $\omega$. In addition, we can choose $f$ such that $[\alpha] \in H^1(M, \mathbb{Z})$, which proves (iii). Finally, if $[\gamma] \neq 0$ then there exists a cycle $z \in H_1(M)$ such that $z \cdot [\gamma] = 1$; choosing $f$ non-negative we obtain $\int_z \alpha \neq 0$, thus $[\alpha] \neq 0$. 


Now Proposition 12 follows from Theorem 16. What is more, the same theorem shows that Proposition 12 is not a criterion:

**Counterexample 17.** The converse to Proposition 12 is not true for manifolds with $b'_1(M) > 1$; see Section 2.3. Indeed, by (6) there exists a Morse form $\omega$ on $M$ such that $c(\omega) = b'_1(M)$. By Theorem 16 there exists a form $\alpha, 0 \neq [\alpha] \in H^1(M, \mathbb{Z})$, such that $\alpha \wedge \omega = 0$ and thus $[\alpha] \sim [\omega] = 0$. The same foliation $\mathcal{F}_\omega$ can be defined by a form of rank $b'_1(M)$ [6, Theorem 4.1], so we can assume that $\rho(\omega) = b'_1(M) > 1$. Then there exists a form $\omega'$ with a minimal foliation and $[\omega'] = [\omega] [1]$; in particular, $[\alpha] \sim [\omega'] = 0$.

Recall that $c(\omega) = \rho H_\omega$ is the total number of homologically independent compact leaves of $\mathcal{F}_\omega$. Theorem 16 states that $c(\omega) \neq 0$ iff there is a suitable $[\alpha] \neq 0$. This can be easily generalized to an arbitrary number $k$: $c(\omega) \geq k$ iff there are $k$ independent $\alpha$‘s, which gives a criterion for existence of $k$ homologically independent compact leaves:

**Theorem 18.** The following conditions are equivalent:

(i) $\mathcal{F}_\omega$ has $k$ homologically independent compact leaves $\gamma_i$;

(ii) There exist $k$ cohomologically independent smooth closed 1-forms $\alpha_i$, $[\alpha_i] \in H^1(M, \mathbb{Z})$, collinear with $\omega$.

If the above conditions hold for $k = b'_1(M)$ then $\mathcal{F}_\omega$ is compactifiable.

**Proof.** (i) $\Rightarrow$ (ii): For each $\gamma_i$ construct a form $\alpha_i$, $[\alpha_i] \neq 0$, as in Theorem 16. Consider a Poincaré duality map $D$ that satisfies (1) for $\gamma_i$. Since $\int_{D\gamma_i} \alpha_j = \delta_{ij}$, all $[\alpha_i]$ are independent.

(ii) $\Rightarrow$ (i): As has been noted in Lemma 15, $\rho H_\alpha$ is determined by $DH_\alpha$.

By Proposition 14 we have $H_\alpha \subseteq H_\omega$ and thus the rank of the whole system $\langle [\alpha_1], \ldots, [\alpha_k] \rangle$ is determined by $H_\omega$, which implies $c(\omega) = \rho H_\omega \geq k$.

Finally, by (5), $c(\omega) \geq k = b'_1(M)$ implies $m(\omega) = 0$, i.e. $\mathcal{F}_\omega$ is compactifiable. \[\square\]

**References**


