On compact leaves of a Morse form foliation

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Abstract. On a compact oriented manifold without boundary, we consider a closed 1-form with singularities of Morse type, called Morse form. We give criteria for the foliation defined by this form to have a compact leaf, to have \( k \) homologically independent compact leaves, and to have no minimal components.

1. Introduction and the results

Consider a compact oriented connected smooth \( n \)-dimensional manifold \( M \) without boundary. On \( M \), consider a smooth differential 1-form \( \omega \) that is closed, i.e., \( d\omega = 0 \). By the Poincaré lemma, it is locally the differential of a function: \( \omega = df \).

In this paper, we assume \( f \) to be a Morse function; then \( \omega \) is called a Morse form. By Morse functions we mean smooth functions with non-degenerate singularities. They are generic (typical) smooth functions: their set is open and dense in the space of smooth functions [7]. Likewise, Morse forms are generic (typical) closed 1-forms: their set is open and dense in the space of all closed 1-forms on \( M \).

Let \( \omega \) be a Morse form on \( M \). The set of its singularities \( \text{Sing} \omega = \{ x \in M \mid \omega_x = 0 \} \) is finite. On \( M \setminus \text{Sing} \omega \) the form \( \omega \) defines a foliation \( \mathcal{F}_\omega \) constructed as follows: For any \( x \in M \setminus \text{Sing} \omega \), the equation \( \omega_x(\xi) = 0 \) defines a distribution of the tangent bundle \( T_x M \). Since \( \omega \) is closed, this distribution is integrable; its integral surfaces are leaves of \( \mathcal{F}_\omega \).

A foliation is a way of slicing the manifold into disjoint submanifolds (called leaves) of lower dimension, in our case the dimension \( n-1 \). This notion is widely

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used in physics. For example, the phase space of a mechanical system is foliated by its energy levels. Foliations of space-time into three-dimensional space-like hypersurfaces have been found to completely characterize the topology of space-time, the singularities describing the topological structure of the gravitational singularities [10].

A foliation $\mathcal{F}_\omega$ has three types of leaves: compact, non-compact compactifiable and non-compact non-compactifiable. If a leaf $\gamma$ is compactified by $\text{Sing}_\omega$, i.e., $\gamma \cup \text{Sing}_\omega$ is compact, then it is called compactifiable, otherwise it is called non-compactifiable. In particular, compact leaves are compactifiable. A foliation is called compactifiable if it has only compactifiable leaves, i.e., if it has no minimal components (areas covered by non-compactifiable leaves).

Existence of compact leaves and existence of non-compactifiable leaves in a given foliation are classical problems of the foliation theory. We consider both these problems for a Morse form foliation.

Denote by $H_\omega \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$, and by $\mathcal{H} \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $\cdot$ is the cycle intersection and $[\omega] : H_1(M) \to \mathbb{R}$ the integration map. We denote $\text{rk}_\omega \equiv \text{rk}_\mathbb{Q} \text{im}[\omega]$.

Melnikova [9] has shown that on a two-dimensional manifold, a foliation $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} \geq \text{rk}\omega - 1$. We generalize this fact to arbitrary dimension and give a stronger formulation: $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} = \text{rk}\omega$ (Theorem 8). In Theorem 8 we also show that $\text{rk}\mathcal{H} \leq \text{rk}\omega$, but $\text{rk}\mathcal{H} \neq \text{rk}\omega - 1$.

Farber et al. [2], [3] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form. We show that this condition is not a criterion. Then we generalize it to arbitrary (not necessarily transitive) Morse forms and improve it to a criterion.

For this, we introduce the notion of collinearity of forms: we call a (not necessarily Morse) smooth closed 1-form $\alpha$ collinear with $\omega$ if $\alpha \wedge \omega = 0$; foliations of collinear forms share entire leaves (Proposition 14). We give a criterion for existence of compact leaves: $\mathcal{F}_\omega$ has a compact leaf iff there exists a form $\alpha \neq 0$ collinear with $\omega$ such that $[\alpha] \in H^1(M, \mathbb{Z})$ (Theorem 16); what is more, $\mathcal{F}_\omega$ has $k$ homologically independent compact leaves iff there exist $k$ cohomologically independent such forms (Theorem 18).

Finally, we give a condition for compactifiability of $\mathcal{F}_\omega$ in terms of existence of a sufficient number of cohomologically independent forms collinear with $\omega$ (Theorem 18).

The paper is organized as follows. In Section 2, we introduce necessary definitions and facts about Morse form foliations. In Section 3, we prove a criterion
for a Morse form foliation to be compactifiable. Finally, in Section 4 we introduce a notion of collinearity of 1-forms and use it to give criteria for a foliation to have a compact leaf or \(k\) homologically independent compact leaves.

2. Definitions and useful facts

Recall that \(M\) is a compact oriented connected smooth \(n\)-dimensional manifold without boundary.

2.1. Poincaré duality map. We call an injection \(D : H_{n-1}(M) \hookrightarrow H_1(M)\) a Poincaré duality map if there exists a basis \(z_i \in H_{n-1}(M)\) such that

\[
z_i \cdot Dz_j = \delta_{ij},
\]

where \(\cdot\) is the intersection form. For any basis \(z_i \in H_{n-1}(M)\), there exists a Poincaré duality map satisfying (1). Obviously, if a subgroup \(G \subseteq H_{n-1}(M)\) is a direct summand in \(H_{n-1}(M)\), i.e. \(H_{n-1}(M) = G \oplus G'\) for some \(G'\), then for any basis \(z_i \in G\) there exists a Poincaré duality map satisfying (1).

Note that for any subgroup \(G \subseteq H_{n-1}(M)\) we have an isomorphism \(DG \cong G\); in particular, \(\text{rk } DG = \text{rk } G\).

2.2. A Morse form foliation. Recall that for a Morse form \(\omega\), the set \(\text{Sing } \omega\) is finite since the singularities are isolated and \(M\) is compact; on \(M \setminus \text{Sing } \omega\) the form defines a foliation \(\mathcal{F}_\omega\). The number of its non-compact compactifiable leaves is finite, since each singularity can compactify no more than four leaves. The union of all non-compactifiable leaves is open and has a finite number \(m(\omega)\) of connected components [1] called minimal components; we call compactifiable a foliation that has no minimal components.

For a compact leaf \(\gamma\) there exists an open neighborhood consisting solely of compact leaves: indeed, integrating \(\omega\) gives near \(\gamma\) a function \(f\) with \(df = \omega\). Hence, the union of all compact leaves is open. Denote by \(H_\omega \subseteq H_{n-1}(M)\) a group generated by all compact leaves of \(\mathcal{F}_\omega\). A Morse form foliation defines the following decomposition [4]:

\[
H_1(M) = DH_\omega \oplus i_*H_1(\Delta),
\]

where \(\Delta\) is the union of all non-compact leaves and singularities, \(i : \Delta \hookrightarrow M\), and \(D : H_{n-1}(M) \to H_1(M)\) is a Poincaré duality map.
The value $c(\omega) = \text{rk} H_\omega$ is the number of homologically independent compact leaves, i.e. $H_\omega$ has a basis of homology classes of compact leaves, $H_\omega = \langle \{\gamma_1, \ldots, \gamma_c(\omega)\} \rangle$ [4]. For a compactifiable foliation, (2) gives

$$c(\omega) \geq \text{rk} \omega,$$

where $\text{rk} \omega = \text{rk} \text{im}[\omega]$, with $[\omega] : H_1(M) \to \mathbb{R}$ being the integration map. Obviously,

$$\text{rk} \omega + \text{rk} \ker[\omega] = b_1(M),$$

the first Betti number.

### 2.3. Non-commutative Betti number

Arnoux and Levitt [1] denoted by $b'_1(M)$ the non-commutative Betti number — the maximal rank (number of free generators) of a free quotient group of $\pi_1(M)$; note that $b'_1(M) \leq b_1(M)$ [8].

**Example 1.** For an $n$-dimensional torus we have $b'_1(T^n) = 1$; for the connected sum $\#$ of direct products $S^1 \times S^n$, $n > 1$, we have $b'_1\left(\bigoplus_{i=1}^p (S^1 \times S^n)\right) = p$; for a genus $g$ two-dimensional surface we have $b'_1(M^2_g) = g$ [5].

The topology of the foliation is connected with $b'_1(M)$ [5]:

$$c(\omega) + m(\omega) \leq b'_1(M),$$

where $c(\omega)$ is the number of homologically independent compact leaves and $m(\omega)$ the number of minimal components.

Denote by $h(M)$ the maximum number of cohomologically independent cocycles $u_i \in H^1(M, \mathbb{Z})$ such that the cup-product $u_i \cup u_j = 0$ [4]. Then $c(\omega) \leq h(M)$ [6, Theorem 3.2] and for some Morse form $\omega$ on $M$ [5, Theorem 8] it holds

$$c(\omega) = b'_1(M),$$

which gives

$$b'_1(M) \leq h(M).$$

### 3. Conditions for compactifiability

Denote by $H \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $H_\omega \subset H_{n-1}(M)$ is the subgroup generated by all compact leaves of $\mathcal{F}_\omega$ and $\cdot$ is the cycle intersection.

**Lemma 2.** It holds:
(i) $H_\omega$ is a direct summand in $H_{n-1}(M)$;
(ii) $H$ is a direct summand in $H_\omega$;
(iii) $H$ is a direct summand in $H_{n-1}(M)$.

**Proof.** It is easy to show that a subgroup of a finitely-generated free abelian group is a direct summand if its quotient is torsion-free.

(i) Let us show that the quotient $H_{n-1}(M)/H_\omega$ is torsion-free. It has been shown in [4] that there exist compact leaves $\gamma_1, \ldots, \gamma_\ell(\omega) \in \mathcal{F}_\omega$ and closed curves $\alpha_1, \ldots, \alpha_\ell(\omega) \subset M$ such that $[\gamma_i]$ form a basis of $H_{\omega}$ and $[\gamma_i] \cdot [\alpha_j] = \delta_{ij}$. Suppose there exists $0 \neq z = z_0 + H_\omega \in H_{n-1}(M)/H_\omega$ such that $kz = 0$ for some $0 \neq k \in \mathbb{Z}$, i.e., $z_0 \notin H_\omega$ but $kz_0 \in H_\omega$. Then $kz_0 = \sum n_i[\gamma_i]$ and $kz_0 \cdot [\alpha_j] = n_j$. Consider $z_1 = \sum \frac{k}{n_i} [\gamma_i] \in H_\omega$, then $kz_1 = kz_0$. Since $H_\omega \subseteq H_{n-1}(M)$ is torsion-free, we obtain $z_0 = z_1 \in H_\omega$; a contradiction.

(ii) Let us show now that the quotient $H_\omega/H$ is torsion-free. Similarly, suppose $z \notin H$, i.e., $z \cdot \ker[\omega] \neq 0$, then $kz \cdot \ker[\omega] \neq 0$ and thus $kz \notin H$.

(iii) follows from (i) and (ii). \qed

Recall that $D : H_{n-1}(M) \to H_1(M)$ is a Poincaré duality map defined by the cycle intersection. By Lemma 2, for a basis $z_i \in H$ there exists a Poincaré duality map that satisfies (1).

**Lemma 3.** Let $z_i$ be a basis of $H \subset H_{n-1}(M)$ and $D$ a corresponding Poincaré duality map. Then the integrals $\int_D z_i \omega$ are independent over $\mathbb{Q}$.

Indeed, suppose $\sum n_i \int_D z_i \omega = 0$, i.e., $z = \sum n_i D z_i \in \ker[\omega]$; then $n_i = z \cdot z_i = 0$.

**Proposition 4.** If $\text{rk } H \geq \text{rk } \omega - 1$ then $\mathcal{F}_\omega$ is compactifiable.

In fact we will show below that $\text{rk } H \neq \text{rk } \omega - 1$, so the above inequality is equivalent to $\text{rk } H = \text{rk } \omega$.

**Proof.** Consider a basis $z_i \in H$ and a corresponding Poincaré duality map $D$. Denote $L_H = \langle \int_D z_i, \omega \rangle$, a linear space over $\mathbb{Q}$; by Lemma 3, $\dim L_H = \text{rk } H$.

Suppose that there exists a minimal component $U$. Then $\text{rk } \omega|_U \geq 2$, i.e., there exist two cycles $s, u \in i_* H_1(U)$, where $i : U \hookrightarrow M$, with independent periods [8]. Denote $L_U = \langle \int_s \omega, \int_u \omega \rangle$, a linear space over $\mathbb{Q}$; $\dim L_U = 2$.

Let us show that $L_H \cap L_U = 0$. Consider $z = n_s s + n_u u$ such that $\int_z \omega \in L_H$, i.e. $\int_z \omega = \sum n_i \int_D z_i \omega$. Thus $z - \sum n_i D z_i \in \ker[\omega]$. By definition, $H \cdot \ker[\omega] = 0$, so $z_j \cdot (z - \sum n_i D z_i) = 0$. Since $z_j$ are generated by compact leaves while $z \in i_* H_1(U)$; we have $z_j \cdot z = 0$. This gives all $n_j = 0$ and thus $\int_z \omega = 0$.

We have $\text{rk } \omega \geq \dim \langle L_H \cup L_U \rangle = \text{rk } H + 2$; a contradiction. \qed
The following condition in terms of compact leaves is geometrically more visual than Proposition 4:

**Corollary 5.** Let $\mathcal{F}_{\omega}$ have $\text{rk} \omega - 1$ homologically independent compact leaves $\gamma_i$ such that $[\gamma_i] \cdot \ker[\omega] = 0$. Then $\mathcal{F}_{\omega}$ is compactifiable, and there exists another compact leaf $\gamma$ homologically independent from all $\gamma_i$.

**Proof.** By Proposition 4, the foliation is compactifiable. Then (3) gives $c(\omega) \geq \text{rk} \omega$, so there exists a compact leaf $\gamma$ such that $[\gamma] \notin ([\gamma_i])$. □

Corollary 5 is not a criterion:

**Counterexample 6.** On a two-dimensional genus 4 surface $M^2_4$ represented as a connected sum of four tori $T^2$, consider a compactifiable foliation such that $\int z_1 \omega = \int z_2 \omega = 1$ and $\int z_3 \omega = \int z_4 \omega = \sqrt{2}$, so that $\text{rk} \omega = 2$ and $c(\omega) = 4$; see Figure 1. Then $(z_1 - z_2), (z_3 - z_4) \in \ker[\omega]$, but for any homologically non-trivial compact leaf $\gamma \in \mathcal{F}_{\omega}$ we have either $[\gamma] \cdot (z_1 - z_2) \neq 0$ or $[\gamma] \cdot (z_3 - z_4) \neq 0$, so there are no $\text{rk} \omega - 1 = 1$ homologically independent leaves such that $[\gamma] \cdot \ker \omega = 0$. Note that still $\text{rk} \mathcal{H} = 2$, cf. Proposition 4.

![Figure 1. A foliation on a connected sum $M^2_4 = \#_4 T^2$.](image)

However, with an additional condition the converse to Corollary 5 is true:

**Proposition 7.** If $\mathcal{F}_{\omega}$ is compactifiable and $\text{rk} \omega = c(\omega)$, then there exist $\text{rk} \omega$ homologically independent compact leaves $\gamma_i \in \mathcal{F}_{\omega}$ such that $[\gamma_i] \cdot \ker[\omega] = 0$. 
Proof. Consider a basis $[\gamma_i] \in H_\omega$. For a compactifiable foliation, $\Delta$ mentioned in (2) is the union of a finite number of compactifiable leaves and singularities, so $i_*H_1(\Delta) \subseteq \ker[\omega]$ and $\rk \omega$ is determined by $D[\gamma_i]$. Since $\rk(D[\gamma_i]) = \rk H_\omega = c(\omega) = \rk \omega$, all corresponding integrals are rationally independent, so $\ker[\omega] = i_*H_1(\Delta)$. Then $\gamma_i \cdot \ker[\omega] = 0$ since $\gamma_i \cap \Delta = \emptyset$. 

In the rest of this section we will study $\rk \mathcal{H}$. By (5), $\rk \mathcal{H} \leq b_1(M)$. The following properties of $\rk \mathcal{H}$ are connected with $\rk \omega$:

**Theorem 8.** It holds:

(i) $\rk \mathcal{H} \leq \rk \omega$.

(ii) $\rk \mathcal{H} \neq \rk \omega - 1$.

(iii) $\mathcal{F}_\omega$ is compactifiable iff $\rk \mathcal{H} = \rk \omega$.

**Proof.** (i) follows from Lemma 3.

(ii) follows from Proposition 4 and (iii).

(iii) If $\rk \mathcal{H} = \rk \omega$ then $\mathcal{F}_\omega$ is compactifiable by Proposition 4.

Let now $\mathcal{F}_\omega$ be compactifiable. By Lemma 2 there exists a Poincaré duality map $D$ that satisfies (1) for a basis $[\gamma_i]$ of $H_\omega$. Consider $\varphi : H_\omega \to \mathbb{R}$, $\varphi(z) = \int_{Dz} \omega$. Since $\Delta$ in (2) consists of a finite number of compactifiable leaves and singularities, we have $i_*H_1(\Delta) \subseteq \ker[\omega]$; in particular, $\rk \omega = \rk \im \varphi$.

Recall that $\mathcal{H} = \{z \in H_\omega \mid z \cdot \ker[\omega] = 0\}$. Let $u = u_1 + u_2 \in H_1(M)$, $u_1 \in DH_\omega$, $u_2 \in i_*H_1(\Delta)$ according to (2). Since $H_\omega \cdot i_*H_1(\Delta) = 0$, we have $z \cdot u = z \cdot u_1$. For a compactifiable foliation, $u_2 \in \ker[\omega]$, so $u \in \ker[\omega]$ iff $u_1 \in \ker[\omega]$. Thus the above definition can be rewritten as $\mathcal{H} = \{z \in H_\omega \mid z \cdot DH_0 = 0\}$, where $H_0 = \ker \varphi$; in other words, $\mathcal{H}$ is the set of all $z = \sum n_i[\gamma_i]$ such that for all $z_k = \sum m_{kj}[\gamma_j]$ that generate $H_0 \subseteq H_\omega$ it holds $z \cdot Dz_k = 0$, i.e., $\sum n_i m_{ki} = 0$.

The latter linear system implies $\rk \mathcal{H} = \rk H_\omega - \rk H_0$. Since $\rk H_\omega = c(\omega)$ and $\rk H_0 = \rk \ker \varphi = \rk H_\omega - \rk \im \varphi = c(\omega) - \rk \omega$, we obtain $\rk \mathcal{H} = \rk \omega$.

Let us consider some special cases.

**Corollary 9.** Let $\ker[\omega] = 0$. Then $\mathcal{F}_\omega$ is compactifiable iff $c(\omega) = b_1(M)$, the first Betti number. In this case the cup-product $\cdot : H^1(M, \mathbb{Z}) \times H^1(M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ is trivial.

**Proof.** Since $\ker[\omega] = 0$, we have $\rk \omega = b_1(M)$ and $\mathcal{H} = H_\omega$. Then the condition $\rk \mathcal{H} = \rk \omega$ from Theorem 8 (iii) is equivalent to $c(\omega) = b_1(M)$ and thus $H_\omega = H_{n-1}(M)$. Then $H^1(M, \mathbb{Z}) = \langle z_i \rangle$, where $z_i$ are cocycles dual to $[\gamma_i]$, a basis of $H_\omega$, and $\gamma_i \cap \gamma_j = \emptyset$ implies $z_i \sim z_j = 0$. 

\[ \square \]
So if \( \ker[\omega] = 0 \) and \( \sim \neq 0 \), then \( F_\omega \) has a minimal component. If, however, \( \sim \equiv 0 \), then both cases are possible. Indeed, on the one hand, in any cohomology class \([\omega]\), \( \text{rk} \omega > 1 \), there exists a Morse form with minimal foliation [1]. On the other hand, the foliation can be compactifiable:

**Example 10.** Consider a connected sum \( M = \bigsqcup_{i=1}^p (S^1 \times S^n) \), \( n > 1 \); see Figure 2. Then \( b_1(M) = b'_1(M) = p \) (Example 1), which by (7) gives \( \sim \equiv 0 \). Consider \( \omega \) given on each \( (S^1 \times S^n) \), by \( \omega_i = \alpha_i dt \), where \( t \) is a coordinate on \( S^1 \) and all \( \alpha_i \in \mathbb{R} \) are independent over \( \mathbb{Q} \) so that \( \text{rk} \omega = p \). Obviously, \( F_\omega \) is compactifiable (its compact leaves are spheres \( S^n \)).

![Figure 2. A foliation on a connected sum \((S^1 \times S^n) \bigsqcup (S^1 \times S^n)\).](image)

**Corollary 11.** For a two-dimensional genus \( g \) surface \( M^2_g \) it holds
\[
\text{rk} \mathcal{H} \leq \text{rk} \omega \leq 2g - c(\omega) \leq 2g - \text{rk} \mathcal{H}.
\] (8)
If \( \text{rk} \mathcal{H} = g \), then \( F_\omega \) is compactifiable.

**Proof.** The lower bound is by Theorem 8 (i). Since leaves are one-dimensional, \( \mathcal{H} \subseteq H_\omega \subseteq \ker[\omega] \) and \( \text{rk} \ker[\omega] = 2g - \text{rk} \omega \) gives the upper bound. If \( \text{rk} \mathcal{H} = g \) then (8) implies \( \text{rk} \omega = g \) and \( F_\omega \) is compactifiable by Theorem 8 (iii).

4. **Criterion for the presence of compact leaves**

Farber *et al.* proved a necessary condition of existence of a compact leaf \( \gamma \) in terms of zero cup-product:

**Proposition 12** ([2, Proposition 9.14],[3, Proposition 3]). For so-called transitive Morse forms, if \( F_\omega \) has a compact leaf with \([\gamma] \neq 0\) then there exists a smooth closed 1-form \( \alpha \), \( 0 \neq [\alpha] \in H^1(M, \mathbb{Z}) \), such that \( [\alpha] \sim [\omega] = 0 \).
The converse is, however, not true; see Counterexample 17 below. Moreover, no sufficient conditions for existing of a compact leaf can be given in cohomolo-
gous terms: any cohomology class \([\omega], \text{rk} \omega > 1\), contains a form with minimal foliation [1].

We call 1-forms \(\alpha\) and \(\beta\) collinear if \(\alpha \wedge \beta = 0\). Using the notion of collinearity instead of zero cup-product, we will generalize Proposition 12 to an arbitrary (not necessarily transitive) Morse form and refine it to a criterion. For closed 1-forms the equation \(\alpha \wedge \beta = 0\) implies \([\alpha] \sim [\beta] = 0\) but not vice versa, so collinearity is a stronger condition.

Denote \(\text{Supp} \alpha = M \setminus \text{Sing} \alpha\). If \(\alpha\) is closed, on \(\text{Supp} \alpha\) the integrable distribution \(\{\alpha = 0\}\) defines a foliation \(F_\alpha\).

**Lemma 13.** For closed collinear 1-forms \(\alpha, \beta\), on \(\text{Supp} \beta\) it holds \(\alpha = f(x) \beta\), where \(f(x)\) is constant on leaves of \(F_\beta\). In particular, on \(\text{Supp} \alpha \cap \text{Supp} \beta\) it holds \(F_\alpha = F_\beta\).

**Proof.** On \(\text{Supp} \beta\) there exists a smooth vector field \(\xi_s\) such that \(\beta(\xi_s) \neq 0\). Consider \(f(x) = \frac{\alpha(\xi_s)}{\beta(\xi_s)}\), which is well-defined: for any vector fields \(\xi_s, \eta_x\) we have \(\alpha(\xi_s)\beta(\eta_x) - \alpha(\eta_x)\beta(\xi_s) = (\alpha \wedge \beta)(\xi_s, \eta_x) = 0\). Thus on \(\text{Supp} \beta\) we have \(\alpha = f(x) \beta\).

Since \(\alpha\) and \(\beta\) are closed, \(df \wedge \beta = da - f d\beta = 0\). Consider a vector field \(\xi\) tangent to the leaves of \(F_\beta\) and \(\eta\) normal to the leaves. Then \(df \wedge (\beta, \eta) = 0\) implies \((df)(\xi) = 0\), i.e. \(f\) is constant on leaves.

**Proposition 14.** Let \(\alpha\) be a smooth closed 1-form collinear with a Morse form \(\omega\); \(\alpha \neq 0\) and \([\alpha] \in H^1(M, \mathbb{Z})\). Then \(\text{Supp} \alpha\) is the union of a non-empty subset of compact leaves of \(F_\omega\) and a subset of compactifiable leaves of \(F_\omega\). These leaves of \(F_\omega\) are leaves of \(F_\alpha\).

**Proof.** All leaves of \(F_\alpha\) are closed. Indeed, since \([\alpha] \in H^1(M, \mathbb{Z})\), it defines a smooth map \(F_{[\alpha]} : M \to S^1\),

\[ F_{[\alpha]}(x) = e^{2\pi i} f_{x_o} \alpha. \]

Obviously, \(F_{[\alpha]}\) is constant on leaves of \(F_\alpha\) and the critical set of \(F_{[\alpha]}\) coincides with \(\text{Sing} \alpha\). So on \(\text{Supp} \alpha\) the map is regular and by the implicit function theorem each leaf of \(F_\alpha\) (which is a connected component of a level \(F_{[\alpha]}^{-1}(y)\), \(y \in S^1\)) is a closed codimension-one submanifold of \(\text{Supp} \alpha\) (not necessarily closed in \(M\)).

Next, if for a leaf \(\gamma \in F_\omega\) it holds \(\gamma \cap \text{Supp} \alpha \neq \emptyset\) then \(\gamma \subseteq \text{Supp} \alpha\). Indeed, suppose there exists \(x_0 \in \gamma \cap \text{Sing} \alpha\). By Lemma 13, on \(\text{Supp} \omega\) it holds \(\alpha = f(x) \omega\),
There exists a smooth closed 1-form \( \gamma \) if \( \Delta \) consists of isolated points, such a leaf \( \gamma \) would be a point. This gives \( \text{Supp} \alpha \cap \text{Sing} \omega = \emptyset \) and thus \( \text{Supp} \alpha \subseteq \text{Supp} \omega \).

Now Lemma 13 implies that all leaves of \( \mathcal{F}_\alpha \) are leaves of \( \mathcal{F}_\omega \). Since all leaves of \( \mathcal{F}_\alpha \) are closed in \( \text{Supp} \alpha \), the latter cannot contain any non-compactifiable leaves of \( \mathcal{F}_\omega \). It cannot consist solely of non-compact compactifiable leaves of \( \mathcal{F}_\omega \) since their number is finite while \( \text{Supp} \alpha \) is open. Thus it must contain compact leaves of \( \mathcal{F}_\omega \).

**Lemma 15.** In the conditions of Proposition 14, if \( [\alpha] \neq 0 \) then \( \mathcal{F}_\alpha \) has a compact leaf with \([\gamma] \neq 0\).

**Proof.** Following the reasoning of [4] it is easy to show that (2) holds for \( \alpha \) even though it is not a Morse form. Since its \( \Delta \) consists of \( \text{Sing} \alpha \) and a finite number of compactifiable leaves, \( \text{rk} \alpha \) is determined by \( DH_\alpha \). However, if \([\gamma] = 0\) for any compact \( \gamma \in \mathcal{F}_\alpha \) then \( H_\alpha = 0 \) and thus \( \text{rk} \alpha = 0 \), i.e., \([\alpha] = 0\).

Now we are ready to prove the main result of this section: a criterion for existence of a compact leaf.

**Theorem 16.** The following conditions are equivalent:

(i) \( \mathcal{F}_\omega \) has a compact leaf \( \gamma \);

(ii) There exists a smooth function \( f(x) \neq \text{const} \) such that \( df \) is collinear with \( \omega \);

(iii) There exists a smooth closed 1-form \( \alpha \neq 0 \), \([\alpha] \in H^1(M, \mathbb{Z})\), collinear with \( \omega \).

Moreover, \( \gamma \) can be chosen with \([\gamma] \neq 0\) iff \( \alpha \) can be chosen with \([\alpha] \neq 0\).

Note that \( f \) and \( \alpha \) are not required to be of Morse type.

**Proof.** (i) \( \Rightarrow \) (ii), (iii): Let \( \gamma \) be a compact leaf. Consider a cylindrical neighborhood \( O(\gamma) = \gamma \times I \) consisting of diffeomorphic leaves. Let \((x^1, \ldots , x^n)\) be local coordinates in \( O(\gamma) \) such that \((x^1, \ldots , x^{n-1})\) are coordinates in \( \gamma \) and \( x^n \) in \( I \). Consider a smooth function \( f(x) = f(x^n) \neq \text{const} \) in \( O(\gamma) \) and \( f(x) = 0 \) on \( M \setminus O(\gamma) \). Let \( x \in O(\gamma) \); consider the leaf \( \gamma' \ni x \). Let \( \eta_1, \eta_2 \in T_x M \); then \( \eta_i = \xi_i + a_i n \), where \( \xi_i \in T_x \gamma' \), \( a_i \in \mathbb{R} \), and \( n \in T_x M \setminus T_x \gamma' \). We obtain \( df(\eta_i) = a_i df(n) \) and \( \omega(\eta_i) = a_i \omega(n) \). Thus \( df \wedge \omega(\eta_1, \eta_2) = 0 \), which proves (ii).

Consider now \( \alpha = f(x) \omega \); obviously, \( \alpha \) is closed and collinear with \( \omega \). In addition, we can choose \( f \) such that \([\alpha] \in H^1(M, \mathbb{Z})\), which proves (iii). Finally, if \([\gamma] \neq 0\) then there exists a cycle \( z \in H_1(M) \) such that \( z \cdot [\gamma] = 1 \); choosing \( f \) non-negative we obtain \( \int_z \alpha \neq 0 \), thus \([\alpha] \neq 0\).
Now Proposition 12 follows from Theorem 16. What is more, the same theorem shows that Proposition 12 is not a criterion: Counterexample 17. The converse to Proposition 12 is not true for manifolds with $b_1'(M) > 1$; see Section 2.3. Indeed, by (6) there exists a Morse form $\omega$ on $M$ such that $c(\omega) = b_1'(M)$. By Theorem 16 there exists a form $\alpha$, $0 \neq [\alpha] \in H^1(M, \mathbb{Z})$, such that $\alpha \wedge \omega = 0$ and thus $[\alpha] \sim [\omega] = 0$. The same foliation $\mathcal{F}_\omega$ can be defined by a form of rank $b_1'(M)$ [6, Theorem 4.1], so we can assume that $\text{rk}_\omega = b_1'(M)$. Then there exists a form $\omega'$ with a minimal foliation and $[\omega'] = [\omega] [1]$; in particular, $[\alpha] \sim [\omega'] = 0$.

Recall that $c(\omega) = \text{rk}_\omega$ is the total number of homologically independent compact leaves of $\mathcal{F}_\omega$. Theorem 16 states that $c(\omega) \neq 0$ iff there is a suitable $[\alpha] \neq 0$. This can be easily generalized to an arbitrary number $k$: $c(\omega) \geq k$ iff there are $k$ independent $\alpha$’s, which gives a criterion for existence of $k$ homologically independent compact leaves:

**Theorem 18.** The following conditions are equivalent:

(i) $\mathcal{F}_\omega$ has $k$ homologically independent compact leaves $\gamma_i$;

(ii) There exist $k$ cohomologically independent smooth closed 1-forms $\alpha_i$, $[\alpha_i] \in H^1(M, \mathbb{Z})$, collinear with $\omega$.

If the above conditions hold for $k = b_1'(M)$ then $\mathcal{F}_\omega$ is compactifiable.

**Proof.** (i) $\Rightarrow$ (ii): For each $\gamma_i$ construct a form $\alpha_i$, $[\alpha_i] \neq 0$, as in Theorem 16. Consider a Poincaré duality map $D$ that satisfies (1) for $\gamma_i$. Since $\int_{\gamma_i} \alpha_i = \delta_{ij}$, all $[\alpha_i]$ are independent.

(ii) $\Rightarrow$ (i): As has been noted in Lemma 15, $\text{rk}_\alpha$ is determined by $DH_\alpha$. By Proposition 14 we have $H_\alpha \subseteq H_\omega$ and thus the rank of the whole system $<[\alpha_1], \ldots, [\alpha_k]>$ is determined by $H_\omega$, which implies $c(\omega) = \text{rk}_H \omega \geq k$.

Finally, by (5), $c(\omega) \geq k = b_1'(M)$ implies $m(\omega) = 0$, i.e. $\mathcal{F}_\omega$ is compactifiable. □

**References**


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