On compact leaves of a Morse form foliation

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Abstract. On a compact oriented manifold without boundary, we consider a closed 1-form with singularities of Morse type, called Morse form. We give criteria for the foliation defined by this form to have a compact leaf, to have $k$ homologically independent compact leaves, and to have no minimal components.

1. Introduction and announce of results

Consider a compact oriented connected smooth $n$-dimensional manifold $M$ without boundary. On $M$, consider a smooth differential 1-form $\omega$ that is closed, i.e., $d\omega = 0$. By the Poincaré lemma, it is locally the differential of a function: $\omega = df$.

In this paper, we assume $f$ to be a Morse function; then $\omega$ is called a Morse form. By Morse functions we mean smooth functions with non-degenerate singularities. They are generic (typical) smooth functions: their set is open and dense in the space of smooth functions [7]. Likewise, Morse forms are generic (typical) closed 1-forms: their set is open and dense in the space of all closed 1-forms on $M$.

Let $\omega$ be a Morse form on $M$. The set of its singularities $\text{Sing} \, \omega = \{ x \in M \mid \omega_x = 0 \}$ is finite. On $M \setminus \text{Sing} \, \omega$ the form $\omega$ defines a foliation $\mathcal{F}_\omega$ constructed as follows: For any $x \in M \setminus \text{Sing} \, \omega$, the equation $\{ \omega_x (\xi) = 0 \}$ defines a distribution of the tangent bundle $T_x M$. Since $\omega$ is closed, this distribution is integrable; its integral surfaces are leaves of $\mathcal{F}_\omega$.

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A foliation is a way of slicing the manifold into disjoint submanifolds (called leaves) of lower dimension, in our case the dimension $n - 1$. This notion is widely used in physics. For example, the phase space of a mechanical system is foliated by its energy levels. Foliations of space-time into three-dimensional space-like hypersurfaces have been found to completely characterize the topology of space-time, the singularities describing the topological structure of the gravitational singularities [10].

A foliation $\mathcal{F}_\omega$ has three types of leaves: compact, non-compact compactifiable and non-compact non-compactifiable. If a leaf $\gamma$ is compactified by $\text{Sing}_\omega$, i.e., $\gamma \cup \text{Sing}_\omega$ is compact, then it is called compactifiable, otherwise it is called non-compactifiable. In particular, compact leaves are compactifiable. A foliation is called compactifiable if it has only compactifiable leaves, i.e., if it has no minimal components (areas covered by non-compactifiable leaves).

Existence of compact leaves and existence of non-compactifiable leaves in a given foliation are classical problems of the foliation theory. We consider both these problems for a Morse form foliation.

Denote by $H_\omega \subseteq H_{n-1}(M)$ a group generated by all compact leaves of $\mathcal{F}_\omega$, and by $\mathcal{H} \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $\cdot$ is the cycle intersection and $[\omega] : H_1(M) \to \mathbb{R}$ the integration map. We denote $\text{rk}\omega \equiv \text{rk}\ker[\omega]$.

Melnikova [9] has shown that on a two-dimensional manifold, a foliation $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} \geq \text{rk}\omega - 1$. We generalize this fact to arbitrary dimension and give a stronger formulation: $\mathcal{F}_\omega$ is compactifiable iff $\text{rk}\mathcal{H} = \text{rk}\omega$ (Theorem 8). In Theorem 8 we also show that $\text{rk}\mathcal{H} \leq \text{rk}\omega$, but $\text{rk}\mathcal{H} \neq \text{rk}\omega - 1$.

Farber et al. [2, 3] gave a necessary condition for existence of a compact leaf in the foliation defined by a so-called transitive Morse form. We show that this condition is not a criterion. Then we generalize it to arbitrary (not necessarily transitive) Morse forms and improve it to a criterion.

For this, we introduce the notion of collinearity of forms: we call a (not necessarily Morse) smooth closed 1-form $\alpha$ collinear with $\omega$ if $\alpha \wedge \omega = 0$; foliations of collinear forms share entire leaves (Proposition 14). We give a criterion for existence of compact leaves: $\mathcal{F}_\omega$ has a compact leaf iff there exists a form $\alpha \neq 0$ collinear with $\omega$ such that $[\alpha] \in H^1(M, \mathbb{Z})$ (Theorem 16); what is more, $\mathcal{F}_\omega$ has $k$ homologically independent compact leaves iff there exist $k$ cohomologically independent such forms (Theorem 18).

Finally, we give a condition for compactifiability of $\mathcal{F}_\omega$ in terms of existence of a sufficient number of cohomologically independent forms collinear with $\omega$ (Theorem 18).
The paper is organized as follows. In Section 2, we introduce necessary definitions and facts about Morse form foliations. In Section 3, we prove a criterion for a Morse form foliation to be compactifiable. Finally, in Section 4 we introduce a notion of collinearity of 1-forms and use it to give criteria for a foliation to have a compact leaf or \( k \) homologically independent compact leaves.

2. Definitions and useful facts

Recall that \( M \) is a compact oriented connected smooth \( n \)-dimensional manifold without boundary.

2.1. Poincaré duality map. We call an injection \( D : H_{n-1}(M) \hookrightarrow H_1(M) \) a Poincaré duality map if there exists a basis \( z_i \in H_{n-1}(M) \) such that

\[
  z_i \cdot Dz_j = \delta_{ij},
\]

(1)

where \( \cdot \) is the intersection form. For any basis \( z_i \in H_{n-1}(M) \), there exists a Poincaré duality map satisfying (1). Obviously, if a subgroup \( G \subseteq H_{n-1}(M) \) is a direct summand in \( H_{n-1}(M) \), i.e. \( H_{n-1}(M) = G \oplus G' \) for some \( G' \), then for any basis \( z_i \in G \) there exists a Poincaré duality map satisfying (1).

Note that for any subgroup \( G \subseteq H_{n-1}(M) \) we have an isomorphism \( DG \cong G \); in particular, \( \text{rk} DG = \text{rk} G \).

2.2. A Morse form foliation. Recall that for a Morse form \( \omega \), the set \( \text{Sing} \omega \) is finite since the singularities are isolated and \( M \) is compact; on \( M \setminus \text{Sing} \omega \) the form defines a foliation \( F_\omega \). The number of its non-compact compactifiable leaves is finite, since each singularity can compactify no more than four leaves. The union of all non-compactifiable leaves is open and has a finite number \( m(\omega) \) of connected components [1] called minimal components; we call compactifiable a foliation that has no minimal components.

For a compact leaf \( \gamma \) there exists an open neighborhood consisting solely of compact leaves: indeed, integrating \( \omega \) gives near \( \gamma \) a function \( f \) with \( df = \omega \). Hence, the union of all compact leaves is open. Denote by \( H_\omega \subseteq H_{n-1}(M) \) a group generated by all compact leaves of \( F_\omega \). A Morse form foliation defines the following decomposition [4]:

\[
  H_1(M) = DH_\omega \oplus i_* H_1(\Delta),
\]

(2)

where \( \Delta \) is the union of all non-compact leaves and singularities, \( i : \Delta \hookrightarrow M \), and \( D : H_{n-1}(M) \to H_1(M) \) is a Poincaré duality map.
The value $c(\omega) = \text{rk}\, H_\omega$ is the number of homologically independent compact leaves, i.e. $H_\omega$ has a basis of homology classes of compact leaves, $H_\omega = \langle[\gamma_1], \ldots, [\gamma_{c(\omega)}]\rangle$ [4]. For a compactifiable foliation, (2) gives

$$c(\omega) \geq \text{rk}\, \omega,$$

where $\text{rk}\, \omega = \text{rk}\, \text{im}[\omega]$, with $[\omega]: H_1(M) \to \mathbb{R}$ being the integration map. Obviously,

$$\text{rk}\, \omega + \text{rk}\, \ker[\omega] = b_1(M),$$

the first Betti number.

### 2.3. Non-commutative Betti number

Arnoux and Levitt [1] denoted by $b'_1(M)$ the non-commutative Betti number—the maximal rank (number of free generators) of a free quotient group of $\pi_1(M)$; note that $b'_1(M) \leq b_1(M)$ [8].

**Example 1.** For an $n$-dimensional torus we have $b'_1(T^n) = 1$; for the connected sum $\#$ of direct products $S^1 \times S^n$, $n > 1$, we have $b'_1 \left(\#_{i=1}^n (S^1 \times S^n)\right) = p$; for a genus $g$ two-dimensional surface we have $b'_1(M_g^2) = g$ [5].

The topology of the foliation is connected with $b'_1(M)$ [5]:

$$c(\omega) + m(\omega) \leq b'_1(M),$$

where $c(\omega)$ is the number of homologically independent compact leaves and $m(\omega)$ the number of minimal components.

Denote by $h(M)$ the maximum number of cohomologically independent co-cycles $u_i \in H^1(M, \mathbb{Z})$ such that the cup-product $u_i \smile u_j = 0$ [4]. Then $c(\omega) \leq h(M)$ [6, Theorem 3.2] and for some Morse form $\omega$ on $M$ [5, Theorem 8] it holds

$$c(\omega) = b'_1(M),$$

which gives

$$b'_1(M) \leq h(M).$$

### 3. Conditions for compactifiability

Denote by $\mathcal{H} \subseteq H_\omega$ a subgroup of all $z \in H_\omega$ such that $z \cdot \ker[\omega] = 0$, where $H_\omega \subseteq H_{n-1}(M)$ is the subgroup generated by all compact leaves of $\mathcal{F}_\omega$ and $\cdot$ is the cycle intersection.
Lemma 2. It holds:

(i) $H_\omega$ is a direct summand in $H_{n-1}(M)$;
(ii) $\mathcal{H}$ is a direct summand in $H_\omega$;
(iii) $\mathcal{H}$ is a direct summand in $H_{n-1}(M)$.

Proof. It is easy to show that a subgroup of a finitely-generated free abelian group is a direct summand iff its quotient is torsion-free.

(i) Let us show that the quotient $H_{n-1}(M)/H_\omega$ is torsion-free. It has been shown in [4] that there exist compact leaves $\gamma_1, \ldots, \gamma_c(\omega) \in \mathcal{F}_\omega$ and closed curves $\alpha_1, \ldots, \alpha_c(\omega) \subset M$ such that $[\gamma_i]$ form a basis of $H_\omega$ and $[\gamma_i] \cdot [\alpha_j] = 0$. Suppose there exists $0 \neq z = z_0 + H_\omega \in H_{n-1}(M)/H_\omega$ such that $kz = 0$ for some $0 \neq k \in \mathbb{Z}$, i.e., $z_0 \notin H_\omega$. Then $kz_0 = \sum n_i[\gamma_i]$ and $kz_0 \cdot [\alpha_j] = n_j$. Consider $z_1 = \sum n_i[\gamma_i] \in H_\omega$, then $kz_1 = kz_0$. Since $H_\omega \subseteq H_{n-1}(M)$ is torsion-free, we obtain $z_0 = z_1 \in H_\omega$; a contradiction.

(ii) Let us show now that the quotient $H_\omega/\mathcal{H}$ is torsion-free. Similarly, suppose $z \notin \mathcal{H}$, i.e., $z \cdot \ker[\omega] \neq 0$, then $kz \cdot \ker[\omega] \neq 0$ and thus $kz \notin \mathcal{H}$.

(iii) follows from (i) and (ii). \(\square\)

Recall that $D : H_{n-1}(M) \rightarrow H_1(M)$ is a Poincaré duality map defined by the cycle intersection. By Lemma 2, for a basis $z_i \in \mathcal{H}$ there exists a Poincaré duality map that satisfies (1).

Lemma 3. Let $z_i$ be a basis of $\mathcal{H} \subset H_{n-1}(M)$ and $D$ a corresponding Poincaré duality map. Then the integrals $\int_{Dz_i} \omega$ are independent over $\mathbb{Q}$.

Indeed, suppose $\sum n_i \int_{Dz_i} \omega = 0$, i.e., $z = \sum n_i Dz_i \in \ker[\omega]$; then $n_i = z \cdot z_i = 0$.

Proposition 4. If $\text{rk } \mathcal{H} \geq \text{rk } \omega - 1$ then $\mathcal{F}_\omega$ is compactifiable.

In fact we will show below that $\text{rk } \mathcal{H} \neq \text{rk } \omega - 1$, so the above inequality is equivalent to $\text{rk } \mathcal{H} = \text{rk } \omega$.

Proof. Consider a basis $z_i \in \mathcal{H}$ and a corresponding Poincaré duality map $D$. Denote $L_\mathcal{H} = \langle \int_{Dz_i} \omega \rangle$, a linear space over $\mathbb{Q}$; by Lemma 3, $\dim L_\mathcal{H} = \text{rk } \mathcal{H}$.

Suppose that there exists a minimal component $U$. Then $\text{rk } \omega|_U \geq 2$, i.e., there exist two cycles $s, u \in i_*H_1(U)$, where $i : U \hookrightarrow M$, with independent periods [8]. Denote $L_U = \langle \int_s \omega, \int_u \omega \rangle$, a linear space over $\mathbb{Q}$; $\dim L_U = 2$.

Let us show that $L_\mathcal{H} \cap L_U = 0$. Consider $z = n_s s + n_u u$ such that $\int_z \omega \in L_\mathcal{H}$, i.e. $\int_z \omega = \sum n_i \int_{Dz_i} \omega$. Thus $z - \sum n_i Dz_i \in \ker[\omega]$. By definition, $\mathcal{H} \cdot \ker[\omega] = 0$, so $z_j \cdot (z - \sum n_i Dz_i) = 0$. Since $z_j$ are generated by compact leaves while $z \in i_*H_1(U)$; we have $z_j \cdot z = 0$. This gives all $n_j = 0$ and thus $\int_z \omega = 0$. 

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We have \( \operatorname{rk} \omega \geq \dim \langle L_H \cup L_U \rangle = \operatorname{rk} H + 2 \); a contradiction. \( \square \)

The following condition in terms of compact leaves is geometrically more visual than Proposition 4:

**Corollary 5.** Let \( \mathcal{F}_\omega \) have \( \operatorname{rk} \omega - 1 \) homologically independent compact leaves \( \gamma_i \) such that \( [\gamma_i] \cdot \ker[\omega] = 0 \). Then \( \mathcal{F}_\omega \) is compactifiable, and there exists another compact leaf \( \gamma \) homologically independent from all \( \gamma_i \).

**Proof.** By Proposition 4, the foliation is compactifiable. Then (3) gives \( c(\omega) \geq \operatorname{rk} \omega \), so there exists a compact leaf \( \gamma \) such that \( [\gamma] \notin \langle [\gamma_i] \rangle \). \( \square \)

Corollary 5 is not a criterion:

**Counterexample 6.** On a two-dimensional genus 4 surface \( M^2 \) represented as a connected sum of four tori \( T^2 \), consider a compactifiable foliation such that \( \int z_1 \omega = \int z_2 \omega = 1 \) and \( \int z_3 \omega = \int z_4 \omega = \sqrt{2} \), so that \( \operatorname{rk} \omega = 2 \) and \( c(\omega) = 4 \); see Figure 1. Then \( (z_1 - z_2), (z_3 - z_4) \in \ker[\omega] \), but for any homologically non-trivial compact leaf \( \gamma \in \mathcal{F}_\omega \) we have either \( [\gamma] \cdot (z_1 - z_2) \neq 0 \) or \( [\gamma] \cdot (z_3 - z_4) \neq 0 \), so there are no \( \operatorname{rk} \omega - 1 = 1 \) homologically independent leaves such that \( [\gamma] \cdot \ker[\omega] = 0 \).

Note that still \( \operatorname{rk} H = 2 \), cf. Proposition 4.

**Figure 1.** A foliation on a connected sum \( M^2_4 = \#_{i=1}^4 T^2 \).

However, with an additional condition the converse to Corollary 5 is true:

**Proposition 7.** If \( \mathcal{F}_\omega \) is compactifiable and \( \operatorname{rk} \omega = c(\omega) \), then there exist \( \operatorname{rk} \omega \) homologically independent compact leaves \( \gamma_i \in \mathcal{F}_\omega \) such that \( [\gamma_i] \cdot \ker[\omega] = 0 \).
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Proof.\(\) Consider a basis \(\{\gamma_i\} \in H_\omega\). For a compactifiable foliation, \(\Delta\) mentioned in (2) is the union of a finite number of compactifiable leaves and singularities, so \(i_*H_1(\Delta) \subseteq \ker[\omega]\) and \(rk\omega\) is determined by \(D[\gamma_i]\). Since \(rk(D[\gamma_i]) = rkH_\omega = c(\omega) = rk\omega\), all corresponding integrals are rationally independent, so \(\ker[\omega] = i_*H_1(\Delta)\). Then \(\gamma_i \cdot \ker[\omega] = 0\) since \(\gamma_i \cap \Delta = \emptyset\). \(\square\)

In the rest of this section we will study \(rk\mathcal{H}\). By (5), \(rk\mathcal{H} \leq b_1'(M)\). The following properties of \(rk\mathcal{H}\) are connected with \(rk\omega\):

Theorem 8.\(\) It holds:

(i) \(rk\mathcal{H} \leq rk\omega\).
(ii) \(rk\mathcal{H} \neq rk\omega - 1\).
(iii) \(\mathcal{F}_\omega\) is compactifiable iff \(rk\mathcal{H} = rk\omega\).

Proof. (i) follows from Lemma 3.
(ii) follows from Proposition 4 and (iii).
(iii) If \(rk\mathcal{H} = rk\omega\) then \(\mathcal{F}_\omega\) is compactifiable by Proposition 4.

Let now \(\mathcal{F}_\omega\) be compactifiable. By Lemma 2 there exists a Poincaré duality map \(D\) that satisfies (1) for a basis \(\{\gamma_i\}\) of \(H_\omega\). Consider \(\varphi : H_\omega \to \mathbb{R}, \varphi(z) = \int_{Dz}\omega\). Since \(\Delta\) in (2) consists of a finite number of compactifiable leaves and singularities, we have \(i_*H_1(\Delta) \subseteq \ker[\omega]\); in particular, \(rk\omega = rk \im \varphi\).

Recall that \(\mathcal{H} = \{z \in H_\omega \mid z \cdot \ker[\omega] = 0\}\). Let \(u = u_1 + u_2 \in H_1(M), u_1 \in DH_\omega, u_2 \in i_*H_1(\Delta)\) according to (2). Since \(H_\omega \cdot i_*H_1(\Delta) = 0\), we have \(z \cdot u = z \cdot u_1\). For a compactifiable foliation, \(u_2 \in \ker[\omega]\), so \(u \in \ker[\omega]\) iff \(u_1 \in \ker[\omega]\). Thus the above definition can be rewritten as \(\mathcal{H} = \{z \in H_\omega \mid z \cdot DH_0 = 0\}\), where \(H_0 = \ker \varphi\); in other words, \(\mathcal{H}\) is the set of all \(z = \sum u_i[\gamma_i]\) such that for all \(z_k = \sum m_{kj}[\gamma_j]\) that generate \(H_0 \subseteq H_\omega\) it holds \(z \cdot Dz_k = 0\), i.e., \(\sum n_km_{ki} = 0\).

The latter linear system implies \(rk\mathcal{H} = rkH_\omega - rkH_0\). Since \(rkH_\omega = c(\omega)\) and \(rkH_0 = rk \ker \varphi = rkH_\omega - rk \im \varphi = c(\omega) - rk\omega\), we obtain \(rk\mathcal{H} = rk\omega\). \(\square\)

Let us consider some special cases.

Corollary 9. \(\) Let \(\ker[\omega] = 0\). Then \(\mathcal{F}_\omega\) is compactifiable iff \(c(\omega) = b_1(M)\), the first Betti number. In this case the cup-product \(\cup : H^1(M,\mathbb{Z}) \times H^1(M,\mathbb{Z}) \to H^2(M,\mathbb{Z})\) is trivial.

Proof. Since \(\ker[\omega] = 0\), we have \(rk\omega = b_1(M)\) and \(\mathcal{H} = H_\omega\). Then the condition \(rk\mathcal{H} = rk\omega\) from Theorem 8 (iii) is equivalent to \(c(\omega) = b_1(M)\) and thus \(H_\omega = H_{n-1}(M)\). Then \(H^1(M,\mathbb{Z}) = \{z_i\}\), where \(z_i\) are cocycles dual to \(\{\gamma_i\}\), a basis of \(H_\omega\), and \(\gamma_i \cap \gamma_j = \emptyset\) implies \(z_i \cup z_j = 0\). \(\square\)
So if $\ker[\omega] = 0$ and $\sim \neq 0$, then $\mathcal{F}_\omega$ has a minimal component. If, however, $\sim \equiv 0$, then both cases are possible. Indeed, on the one hand, in any cohomology class $[\omega]$, $\text{rk}\omega > 1$, there exists a Morse form with minimal foliation [1]. On the other hand, the foliation can be compactifiable:

**Example 10.** Consider a connected sum $M = \#_{i=1}^{p} (S^1 \times S^n)_i$, $n > 1$; see Figure 2. Then $b_1(M) = b'_1(M) = p$ (Example 1), which by (7) gives $\sim \equiv 0$. Consider $\omega$ given on each $(S^1 \times S^n)_i$ by $\omega_i = \alpha_i dt$, where $t$ is a coordinate on $S^1$ and all $\alpha_i \in \mathbb{R}$ are independent over $\mathbb{Q}$ so that $\text{rk}\omega = p$. Obviously, $\mathcal{F}_\omega$ is compactifiable (its compact leaves are spheres $S^n$).

![Figure 2. A foliation on a connected sum $\mathcal{S}^1 \times S^n$.](image)

**Corollary 11.** For a two-dimensional genus $g$ surface $M^2_g$ it holds

$$\text{rk}\mathcal{H} \leq \text{rk}\omega \leq 2g - c(\omega) \leq 2g - \text{rk}\mathcal{H}. \quad (8)$$

If $\text{rk}\mathcal{H} = g$, then $\mathcal{F}_\omega$ is compactifiable.

**Proof.** The lower bound is by Theorem 8 (i). Since leaves are one-dimensional, $\mathcal{H} \subseteq H_\omega \subseteq \ker[\omega]$ and $\text{rk}\ker[\omega] = 2g - \text{rk}\omega$ gives the upper bound. If $\text{rk}\mathcal{H} = g$ then (8) implies $\text{rk}\omega = g$ and $\mathcal{F}_\omega$ is compactifiable by Theorem 8 (iii). □

4. **Criterion for the presence of compact leaves**

Farber et al. proved a necessary condition of existence of a compact leaf $\gamma$ in terms of zero cup-product:

**Proposition 12 ([2, Proposition 9.14],[3, Proposition 3]).** For so-called transitive Morse forms, if $\mathcal{F}_\omega$ has a compact leaf with $[\gamma] \neq 0$ then there exists a smooth closed 1-form $\alpha$, $0 \neq [\alpha] \in H^1(M, \mathbb{Z})$, such that $[\alpha] \sim [\omega] = 0$. 
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The converse is, however, not true; see Counterexample 17 below. Moreover, no sufficient conditions for existing of a compact leaf can be given in cohomological terms: any cohomology class $[\omega]$, $\text{rk} \omega > 1$, contains a form with minimal foliation [1].

We call 1-forms $\alpha$ and $\beta$ colinear if $\alpha \wedge \beta = 0$. Using the notion of colinearity instead of zero cup-product, we will generalize Proposition 12 to an arbitrary (not necessarily transitive) Morse form and refine it to a criterion. For closed 1-forms the equation $\alpha \wedge \beta = 0$ implies $[\alpha] \sim [\beta] = 0$ but not vice versa, so colinearity is a stronger condition.

Denote $\text{Supp} \alpha = M \setminus \text{Sing} \alpha$. If $\alpha$ is closed, on $\text{Supp} \alpha$ the integrable distribution $\{\alpha = 0\}$ defines a foliation $\mathcal{F}_\alpha$.

**Lemma 13.** For closed colinear 1-forms $\alpha, \beta$, on $\text{Supp} \beta$ it holds $\alpha = f(x)\beta$, where $f(x)$ is constant on leaves of $\mathcal{F}_\beta$. In particular, on $\text{Supp} \alpha \cap \text{Supp} \beta$ it holds $\mathcal{F}_\alpha = \mathcal{F}_\beta$.

**Proof.** On $\text{Supp} \beta$ there exists a smooth vector field $\xi_x$ such that $\beta(\xi_x) \neq 0$. Consider $f(x) = \frac{\alpha(\xi_x)}{\beta(\xi_x)}$, which is well-defined: for any vector fields $\xi_x, \eta_x$ we have $\alpha(\xi_x)\beta(\eta_x) - \alpha(\eta_x)\beta(\xi_x) = (\alpha \wedge \beta)(\xi_x, \eta_x) = 0$. Thus on $\text{Supp} \beta$ we have $\alpha = f(x)\beta$.

Since $\alpha$ and $\beta$ are closed, $df \wedge \beta = d\alpha - f d\beta = 0$. Consider a vector field $\xi$ tangent to the leaves of $\mathcal{F}_\beta$ and $\eta$ normal to the leaves. Then $df \wedge \beta(\xi, \eta) = 0$ implies $(df)(\xi) = 0$, i.e. $f$ is constant on leaves. □

**Proposition 14.** Let $\alpha$ be a smooth closed 1-form colinear with a Morse form $\omega$; $\alpha \neq 0$ and $[\alpha] \in H^1(M, \mathbb{Z})$. Then $\text{Supp} \alpha$ is the union of a non-empty subset of compact leaves of $\mathcal{F}_\omega$ and a subset of compactifiable leaves of $\mathcal{F}_\omega$. These leaves of $\mathcal{F}_\omega$ are leaves of $\mathcal{F}_\alpha$.

**Proof.** All leaves of $\mathcal{F}_\alpha$ are closed. Indeed, since $[\alpha] \in H^1(M, \mathbb{Z})$, it defines a smooth map $F_{[\alpha]} : M \rightarrow S^1$, 

$$F_{[\alpha]}(x) = e^{2\pi i \int_{x_0}^x \alpha}.$$ 

Obviously, $F_{[\alpha]}$ is constant on leaves of $\mathcal{F}_\alpha$, and the critical set of $F_{[\alpha]}$ coincides with $\text{Sing} \alpha$. So on $\text{Supp} \alpha$ the map is regular and by the implicit function theorem each leaf of $\mathcal{F}_\alpha$ (which is a connected component of a level $F_{[\alpha]}^{-1}(y), y \in S^1$) is a closed codimension-one submanifold of $\text{Supp} \alpha$ (not necessarily closed in $M$).

Next, if for a leaf $\gamma \in \mathcal{F}_\omega$ it holds $\gamma \cap \text{Supp} \alpha \neq \emptyset$ then $\gamma \subseteq \text{Supp} \alpha$. Indeed, suppose there exists $x_0 \in \gamma \cap \text{Sing} \alpha$. By Lemma 13, on $\text{Supp} \omega$ it holds $\alpha = f(x)\omega$, 

$$f(x) = e^{2\pi i \int_{x_0}^x \alpha}.$$
where the function $f(x)$ is constant on leaves. Since $x_0 \in \text{Supp} \omega$, we have $f(x_0) = 0$ and so $f|_\gamma = 0$, which gives $\gamma \cap \text{Supp} \alpha = \emptyset$; a contradiction.

Similarly, if for a leaf $\gamma \in F_\alpha$ it holds $\gamma \cap \text{Sing} \omega \neq \emptyset$ then $\gamma \subseteq \text{Sing} \omega$.

However, since $\text{Sing} \omega$ consists of isolated points, such a leaf $\gamma$ would be a point. This gives $\text{Supp} \alpha \cap \text{Sing} \omega = \emptyset$ and thus $\text{Supp} \alpha \subseteq \text{Supp} \omega$.

Now Lemma 13 implies that all leaves of $F_\alpha$ are leaves of $F_\omega$. Since all leaves of $F_\alpha$ are closed in $\text{Supp} \alpha$, the latter cannot contain any non-compactifiable leaves of $F_\omega$. It cannot consist solely of non-compactifiable leaves of $F_\omega$ since their number is finite while $\text{Supp} \alpha$ is open. Thus it must contain compact leaves of $F_\omega$.

**Lemma 15.** In the conditions of Proposition 14, if $[\alpha] \neq 0$ then $F_\alpha$ has a compact leaf with $[\gamma] \neq 0$.

**Proof.** Following the reasoning of [4] it is easy to show that (2) holds for $\alpha$ even though it is not a Morse form. Since its $\Delta$ consists of $\text{Sing} \alpha$ and a finite number of compactifiable leaves, $\text{rk} \alpha$ is determined by $DH_\alpha$. However, if $[\gamma] = 0$ for any compact $\gamma \in F_\alpha$ then $H_\alpha = 0$ and thus $\text{rk} \alpha = 0$, i.e., $[\alpha] = 0$.

Now we are ready to proof the main result of this section: a criterion for existence of a compact leaf.

**Theorem 16.** The following conditions are equivalent:

(i) $F_\omega$ has a compact leaf $\gamma$;

(ii) There exists a smooth function $f(x) \neq \text{const}$ such that $df$ is collinear with $\omega$;

(iii) There exists a smooth closed 1-form $\alpha \neq 0$, $[\alpha] \in H^1(M, \mathbb{Z})$, collinear with $\omega$.

Moreover, $\gamma$ can be chosen with $[\gamma] \neq 0$ iff $\alpha$ can be chosen with $[\alpha] \neq 0$.

Note that $f$ and $\alpha$ are not required to be of Morse type.

**Proof.** (i) $\Rightarrow$ (ii), (iii): Let $\gamma$ be a compact leaf. Consider a cylindrical neighborhood $O(\gamma) = \gamma \times I$ consisting of diffeomorphic leaves. Let $(x^1, \ldots, x^n)$ be local coordinates in $O(\gamma)$ such that $(x^1, \ldots, x^{n-1})$ are coordinates in $\gamma$ and $x^n$ in $I$. Consider a smooth function $f(x) = f(x^n) \neq \text{const}$ in $O(\gamma)$ and $f(x) = 0$ on $M \setminus O(\gamma)$. Let $x \in O(\gamma)$; consider the leaf $\gamma' \ni x$. Let $\eta_1, \eta_2 \in T_x M$; then $\eta_i = \xi_i + a_i n$, where $\xi_i \in T_x \gamma'$, $a_i \in \mathbb{R}$, and $n \in T_x M \setminus T_x \gamma'$. We obtain $df(\eta_i) = a_i df(n)$ and $\omega(\eta_i) = a_i \omega(n)$. Thus $df \wedge \omega(\eta_1, \eta_2) = 0$, which proves (ii).

Consider now $\alpha = f(x) \omega$; obviously, $\alpha$ is closed and collinear with $\omega$. In addition, we can choose $f$ such that $[\alpha] \in H^1(M, \mathbb{Z})$, which proves (iii). Finally, if $[\gamma] \neq 0$ then there exists a cycle $z \in H_1(M)$ such that $z \cdot [\gamma] = 1$; choosing $f$ non-negative we obtain $\int_z \alpha \neq 0$, thus $[\alpha] \neq 0$. 


(ii), (iii) ⇒ (i): This has been shown as Proposition 14 and Lemma 15. □

Now Proposition 12 follows from Theorem 16. What is more, the same theorem shows that Proposition 12 is not a criterion:

Counterexample 17. The converse to Proposition 12 is not true for manifolds with \( b'_1(M) > 1 \); see Section 2.3. Indeed, by (6) there exists a Morse form \( \omega \) on \( M \) such that \( c(\omega) = b'_1(M) \). By Theorem 16 there exists a form \( \alpha, 0 \neq [\alpha] \in H^1(M, \mathbb{Z}) \), such that \( \alpha \wedge \omega = 0 \) and thus \( [\alpha] \sim [\omega] = 0 \). The same foliation \( \mathcal{F}_\omega \) can be defined by a form of rank \( b'_1(M) \) [6, Theorem 4.1], so we can assume that \( \text{rk} \omega = b'_1(M) > 1 \). Then there exists a form \( \omega' \) with a minimal foliation and \( [\omega'] = [\omega] \) [1]; in particular, \( [\alpha] \sim [\omega'] = 0 \).

Recall that \( c(\omega) = \text{rk} H_\omega \) is the total number of homologically independent compact leaves of \( \mathcal{F}_\omega \). Theorem 16 states that \( c(\omega) \neq 0 \) iff there is a suitable \( [\alpha] \neq 0 \). This can be easily generalized to an arbitrary number \( k \): \( c(\omega) \geq k \) iff there are \( k \) independent \( \alpha \)'s, which gives a criterion for existence of \( k \) homologically independent compact leaves:

**Theorem 18.** The following conditions are equivalent:

(i) \( \mathcal{F}_\omega \) has \( k \) homologically independent compact leaves \( \gamma_i \);

(ii) There exist \( k \) cohomologically independent smooth closed 1-forms \( \alpha_i, [\alpha_i] \in H^1(M, \mathbb{Z}) \), collinear with \( \omega \).

If the above conditions hold for \( k = b'_1(M) \) then \( \mathcal{F}_\omega \) is compactifiable.

**Proof.** (i) ⇒ (ii): For each \( \gamma_i \) construct a form \( \alpha_i, [\alpha_i] \neq 0 \), as in Theorem 16. Consider a Poincaré duality map \( D \) that satisfies (1) for \( \gamma_i \). Since \( \int_{\gamma_i} \alpha_j = \delta_{ij} \), all \( [\alpha_i] \) are independent.

(ii) ⇒ (i): As has been noted in Lemma 15, \( \text{rk} \alpha_i \) is determined by \( DH_{\alpha_i} \). By Proposition 14 we have \( H_{\alpha_i} \subseteq H_\omega \) and thus the rank of the whole system \( \langle [\alpha_1], \ldots, [\alpha_k] \rangle \) is determined by \( H_\omega \), which implies \( c(\omega) = \text{rk} H_\omega \geq k \).

Finally, by (5), \( c(\omega) \geq k = b'_1(M) \) implies \( m(\omega) = 0 \), i.e. \( \mathcal{F}_\omega \) is compactifiable. □

**References**


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