Realization of a graph as the Reeb graph of a Morse, Morse–Bott or round function

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Reeb graph R_f of a function $f : M \to \mathbb{R}$ is a topological space obtained by contracting the connected components of the level sets of f to points, endowed with the quotient topology; for a smooth function, connected components containing critical points are called *vertices*, i.e., the *Reeb graph of a smooth function* is the quotient space with marked points.

By a *graph* we understand a pseudograph (allowing loop edges and multiple edges); it has a geometric realization as a one-dimensional CW complex, in which 0-cells correspond to vertices and 1-cells to edges. A graph needs not to be connected.

Definition 1. We say that a Reeb graph R_f *has the structure of a finite graph* G, or R_f *is isomorphic to* G, or R_f *is* G, if there exists a homeomorphism $R_f \to G$ mapping one-to-one the vertices of R_f to the vertices of G.

Generally, the Reeb graph is not a finite graph; in our talk we consider a simple counterexamle. Recently Saeki proved a criterion:

Theorem 2 ([1]). *Let M be a closed manifold, $f : M \to \mathbb{R}$ a smooth function. Then the Reeb graph R_f has the structure of a finite graph if and only if f has a finite number of critical values.*

Every graph without loop edges is the Reeb graph of some function:

Theorem 3 ([2]). *Let G be a finite graph. Then there exist a closed manifold M, and a smooth function $f : M \to \mathbb{R}$ such that its Reeb graph R_f has the structure of G if and only if G has no loop edges.*

The problem of whether a finite graph is the Reeb graph of some function was first studied in 2006 by Sharko [3]. He considered functions with finite critical set $Crit(f)$. In particular, he showed that the graph shown in Figure 1 is not the Reeb graph of any such function.

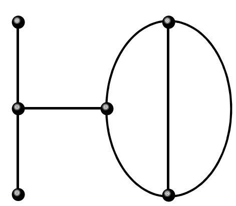

Figure 1.

Below we give criteria for a graph to be the Reeb graph of a function of a given class on a closed manifold: Morse, Morse–Bott, round, and in general smooth functions whose critical set $Crit(f)$ consists of a finite number of submanifolds.

In contrast to works of Michalak [4] and Martínez-Alfaro *et al.* [5] who studied the realization problem in terms of the graph orientation, the following criteria are given in terms of the graph structure, namely, the structure of its *leaf blocks*, i.e., maximal biconnected subgraphs containing at most one cut vertex:

Theorem 4 ([6]). *A graph G is isomorphic to the Reeb graph R_f of some smooth function f with finite $Crit(f)$ on a closed manifold if and only if G is finite, has no loop edges, and all its leaf blocks are path graphs on 2 vertices (closed intervals). The function f can be chosen Morse.*

Theorem 5 ([7]). *For any given $n \geq 2$, a graph G is the Reeb graph R_f of some smooth f whose $Crit(f)$ is a finite number of submanifolds, on closed n-manifold if and only if G is finite, has no loop edges, and each leaf block L has a vertex v with $\deg v \leq 2$, or two such vertices if L is a non-trivial (has an edge) connected component of G. The function f can be chosen Morse–Bott.*

This theorem shows that Sharko's graph in Figure 1 cannot be realized even as the Reeb graph of a function whose $Crit(f)$ is a finite number of submanifolds. Indeed, this graph has three leaf blocks, two of them being closed intervals, and the third leaf block has only 3-vertices.

Morse–Bott functions play a special role in the Reeb graph theory (cf. Theorem 3):

Theorem 6 ([8]). *Any finite graph is homeomorphic to the Reeb graph of a Morse–Bott function.*

Note that, in contrast to Theorem 3, this theorem is true even for graphs with loop edges.

Critical set of a *round function* consists of a finite number of circles. For a round function, the structure of its Reeb graph depends not only on leaf blocks, but also on the dimension of the manifold and its orientability:

Theorem 7 ([7]). *A graph G is isomorphic to the Reeb graph of a round function $f : M^n \to \mathbb{R}$ on a closed n-dimensional manifold if and only if G is finite, has no loop edges, and*

$$\text{each its leaf block} \begin{cases} \text{has a non-cut vertex } v \text{ with } \deg v = 2 & \text{if } n = 2, \text{ orientable surface} \\ \text{has a non-cut vertex } v \text{ with } \deg v \leq 2 & \text{if } n = 2, \text{ non-orientable surface} \\ \text{is a path graph on 2 vertices (closed interval)} & \text{if } n \geq 3. \end{cases}$$

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